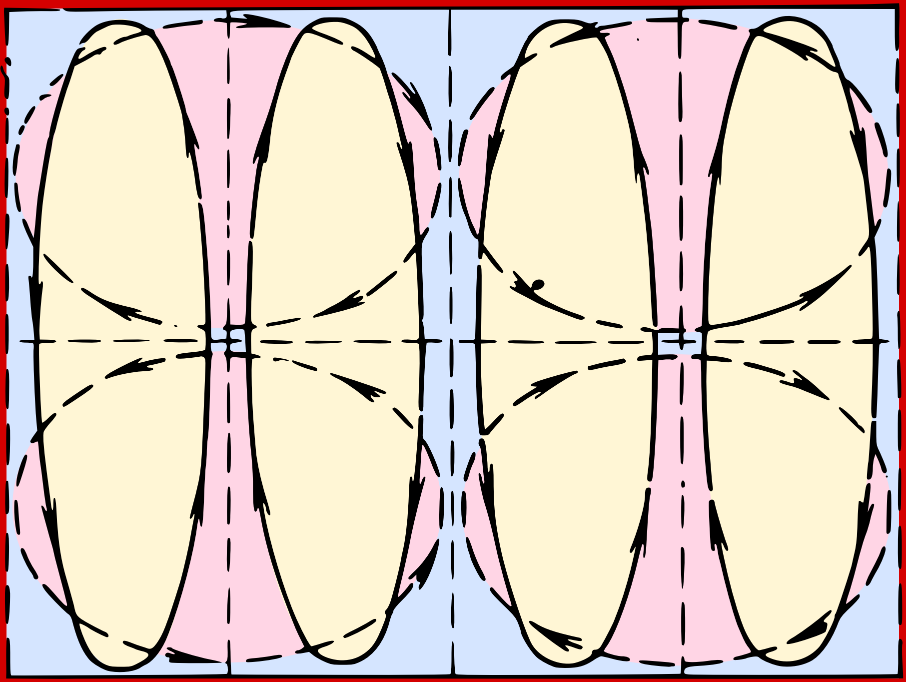
Introduction to Plasma Physics

by B.M. SMIRNOV



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by B. M. Smirnov

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На английском языке

Preface

This book is based on a series of lectures delivered by the author over a ten year period at the Moscow Power-Engineering Institute to technology and chemistry undergraduates not specializing in physics. The book aims at providing a concise yet general description of the physics of weakly ionized plasma so that a budding engineer or chemist can obtain a general understanding of the phenomena occurring in the plasma of a laboratory setting. This understanding is necessary since low-temperature plasma is increasingly used in technology.

The character of the book's intended readership demands that the mathematics of the book be relatively unsophisticated. The author believes that the purpose of this book cannot be achieved merely by including descriptive material and formulas without derivation. This approach can hardly contribute to understanding the subject since the student cannot see all the conditions providing for the validity of the result. We use another approach. The book extensively employs various evaluative techniques, which show the dependence of the result on the parameters of the problem and give its value within order of magnitude. Moreover, for some functions only their limiting values are determined or simple assumptions are made to find these functions. These methods yield a correct qualitative picture of the subject and considerably simplify the discussion. However, the simplicity of discussion is essentially thought-provoking and creates a profound understanding of the subject.

The amount of the material used in the book and the form of its presentation were chosen to provide engineering students with the general knowledge of the fundamentals of the plasma physics, which they will need when working with plasma-containing systems.

The reader is assumed to know the material given in a basic course of general college physics. The choice of problems and their treatment in the book were to some extent prompted by the author's experience in the applied plasma research.

A list of literature for further reading is given at the end of the book.

B. M. Smirnov

Contents

1 Plasma in Nature and in Laboratory Systems

What is plasma? The term "plasma" first appeared in physics in the 1920s. Plasma is a conducting gas, that is, a gas which contains a noticeable proportion of charged particles (electrons and ions). To understand the conditions of plasma formation let us compare a plasma and a mixture of chemically active gases. For instance, the following chemical reaction can occur in the air, which is basically a mixture of nitrogen and oxygen:

$$N_2+O_2 \rightleftarrows 2NO-41.5\ \text{kcal-mol}^{-1} \qquad (1.1)$$

Hence, a small amount of nitric oxide NO is present in the air at the equilibrium between nitrogen and oxygen. According to the Le Chatelier principle, increasing the air temperature results in a larger equilibrium amount of nitric oxide.

The equilibrium between the neutral and charged particles is similar to the above case. An atom or molecule consists of bound positively charged nuclei and negatively charged electrons. At high temperatures the bonds can be broken giving rise to electrons and positively charged ions. For instance, the respective chemical reaction for the nitrogen molecule is

$$N_2 \rightleftarrows N_2^+ + e-360\ \text{kcal-mol}^{-1} \qquad (1.2)$$

The bonding energy of an outer electron in an atom or molecule is roughly ten times the chemical bond energy. Therefore, production of charged particles in this reaction becomes noticeable at temperatures of the order of tens of thousands of degrees. For instance, the sun's photosphere, which emits the main part of the solar radiation and where the temperature is about 6000 K and the hydrogen atom density is of the order of 10^{17} cm^{-3}, has been found to have the ratio between the densities of the charged particles and the neutral particles (the degree of ionization) of about 5×10^{-3}.

The equilibrium density of the charged particles at room temperature is practically zero owing to the high bonding energies of the electrons in atoms and molecules. However, by placing the gas into an electric field, the gas can be made electrically conductive even at room temperature. The electrons become strongly heated when moving in the electric field and receiving energy from it. The electric field does not affect the gas molecules and even if the degree of ionization of the gas is low, the temperature of the molecules remains at the room level. Such a conducting gas in an external electric field is called the gas discharge.

Plasma can be created in different ways. We shall discuss below in brief the principal types of plasma encountered in practice or research. The diagram in Fig. 1 illustrates the parameters of the plasmas found in various systems.

Laboratory equipment for maintaining plasma. The gas discharge is the most popular technique for producing plasma under laboratory conditions. The gas discharge is used for exciting most gas lasers; gas discharge as well as radiation sources and lamps which generate radiation in a wide wavelength range are the basis of plasmatrons. There are various useful applications of the gas discharge.

The gas discharge is a gas space across which a voltage is applied. Charged particles are produced in this space and they move in the electric field and take energy from it. If the charged particles are produced owing to the effect of an external agent, the resulting gas discharge is called nonself-maintained discharge, in contrast to the self-maintained discharge. The principal useful types of the self-maintained discharge are the glow discharge and the arc discharge; the essential difference between them consists in the process of electron production in the cathode's vicinity. The electron density is 10^7-10^{12} cm^{-3} for the glow discharge and higher for the arc discharge.

The density of the charged particles (electrons and ions) in the gas discharge is much lower than the density of the neutral particles (atoms and molecules). This plasma is called weakly ionized or low-temperature plasma since the mean energy of electrons or ions in it is much lower than the ionization potential for the gas particles. Another extreme is the hot plasma where the mean energy of ions is much higher than the ionization potential of the gas particles. Such plasma contains ions and electrons and practically lacks neutral particles.

An example of hot plasma is the thermonuclear plasma, that is, the plasma which must be created for the course of a controlled thermonuclear reaction. The most practicable thermonuclear fusion reaction involves deuterium nuclei or nuclei of deuterium and tritium (the hydrogen isotopes). To make possible this reaction the deuterium or tritium ions must be able to enter the reaction during the time of plasma confinement, that is, when the ions are in the reaction volume.

In the existing laboratory installations, this condition is satisfied at the ion temperature over hundreds of millions degrees and when the product of the density N of charged particles by the plasma confinement time τ exceeds 10^{16} s-cm^{-3}. For the plasma containing deuterium and tritium nuclei, $N\tau$ must be more than 10^{14} s-cm^{-3} (the Lawson criterion). If this condition is met, the self-maintaining thermonuclear reaction can occur when the heat released by the reaction maintains the temperature of the particles needed for continuation of the reaction. The existing laboratory installations do not achieve these values.* However, the thermonuclear plasma is carefully studied and some advance is forseeable in this field.

Note that the energy of radiation from the sun and stars also is produced in thermonuclear fusion reaction involving the hydrogen nuclei, or protons. This reaction is less efficient than the reaction with the deuterium or tritium ions, but since the reaction volume in stars is very large, the temperature proves to be of the order of ten million degrees, rhat is, considerably lower than the temperature of the laboratory thermonuclear plasma for a more favourable reaction.

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* The Tokamak-type installations, which are considered at present to be the most promising ones for controlled thermonuclear fusion, make it possible to reach values of $N\tau$ less than 10^{13} s-cm^{-3} and ion temperature less than 10^7 K [see M. S. Rabinovich, *Fizika plazmy* 1, 335 (1975): "Results of the V International Conference on plasma physics and controlled thermonuclear fusion, Tokyo, 10-15 November 1974"; English translation in *Soviet Journal of Plasma Physics*].

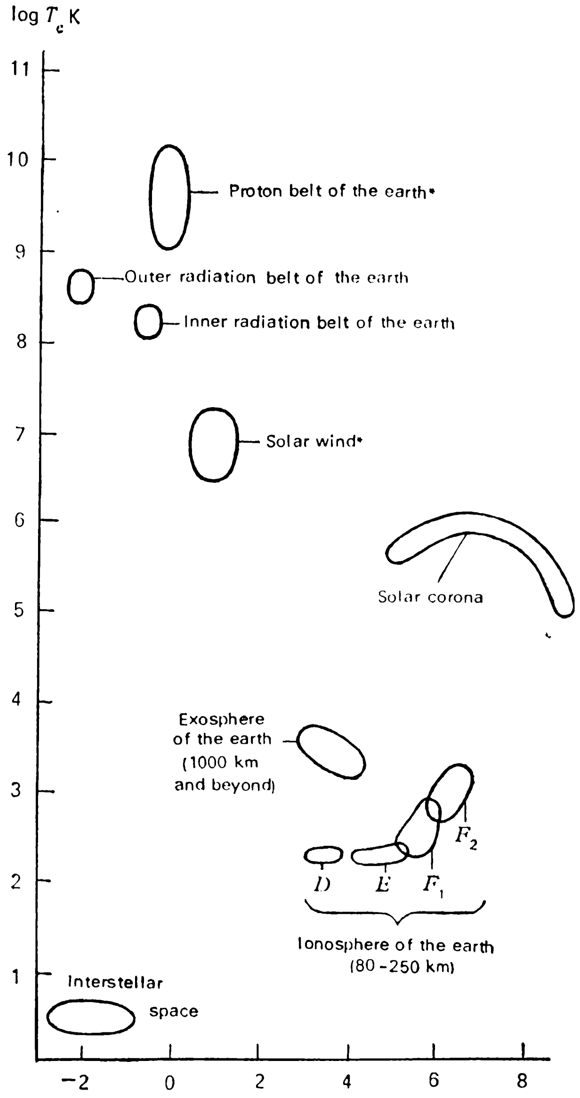
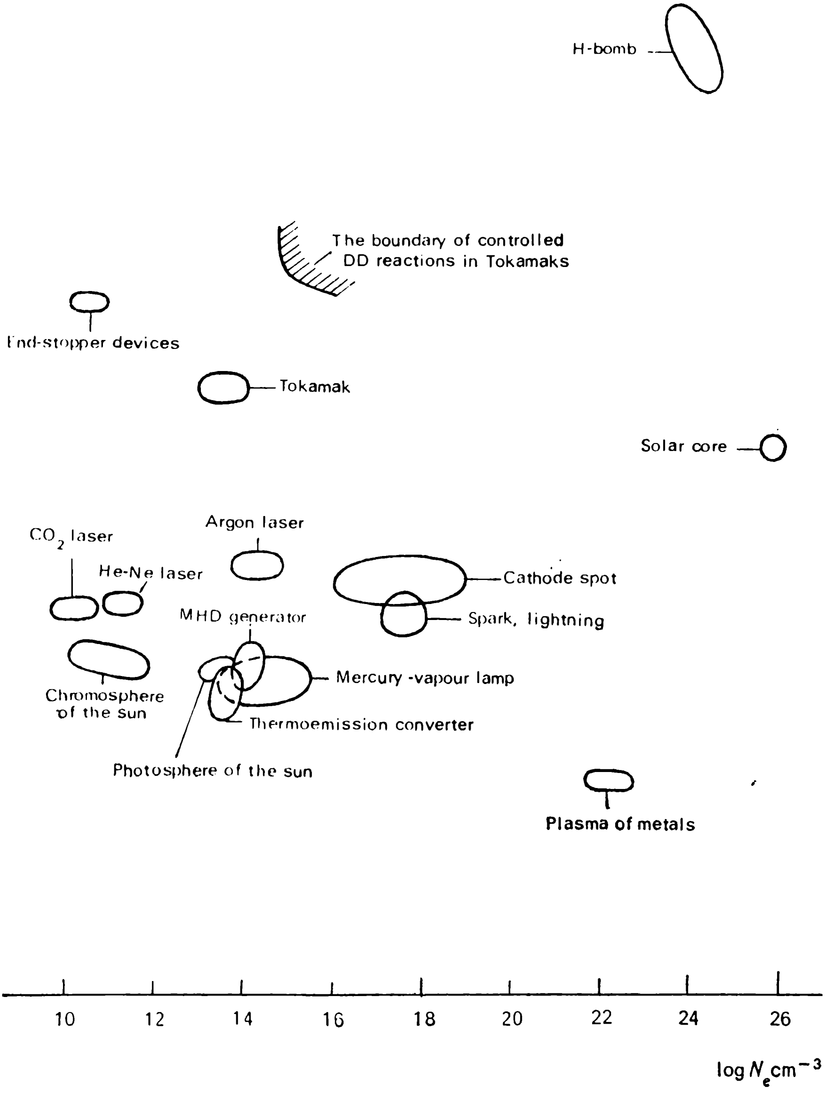


FIG. 1. The plasma parameters: T_e is the electron temperature, and an asterisk denotes plasmas for which the ion temperature T_i is given; N_e is the electron density.


H-bomb
The boundary of controlled DD reactions in Tokamaks
End-stopper devices
Tokamak
Solar core
CO_2 laser
Argon laser
He-Ne laser
Cathode spot
Spark, lightning
MHD generator
Chromosphere of the sun
Mercury-vapour lamp
Thermoemission converter
Photosphere of the sun
Plasma of metals
10
12
14
16
18
20
22
24
26
$\log N_e \mathrm{cm}^{-3}$

The low-temperature plasma is used in the laboratory installations of various types, apart from the gas discharge which can produce plasmas with different parameters. Let us discuss some of these installations. If a magnetic field is applied perpendicular to the flow of weakly ionized gas, an electric current passes perpendicular to the flow and to the magnetic field. If an electric field is applied opposite to this current, we obtain an electric power generator which transforms the kinetic energy of flow into electric energy. Such installations are called the magnetohydrodynamic (MHD) generators.

The greater the magnetic field, the density of the charged particles, and the gas flow velocity, the greater the energy that is produced by the unit volume of the MHD generator. The magnetohydrodynamic transformation of energy is a very promising method since it provides for high power production per unit volume of the installation and high efficiency. There are two types of the MHD generator configuration: open-cycle and closed-cycle. In the MHD generators of the open-cycle type, the working gas passes the conversion volume only once and then is discharged. Application of such MHD generators already has been started at heat power plants where they contribute to increasing the total plant's efficiency. In the closed-cycle MHD generator the working gas repeatedly passes the conversion volume. Practicable closed-cycle MHD generators are still being developed.

If we connect two parallel metal plates with different work functions*, this will give rise to a potential difference across the vacuum gap between the plates. If we then heat one of the plates to a high temperature, there will be some electron emission from it and part of the electrons will reach the cold plate. We shall heat the plate with the higher work function and interconnect the plates via a load. Since the electrons spend energy when passing from one plate to another through the vacuum gap, the electric energy will be liberated at the load. Hence, this system, which is called the thermoemission converter, converts thermal energy into electric energy.

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* The work function is the energy needed by an electron to leave a metal's surface.

The efficiency of the thermoemission converter is low (less than 20%) and for a high temperature of the plates its main advantage is compactness, that is, it produces a high electric power per unit area of the plates. The uncompensated charge of electrons in the gap between the plates gives rise to an electric field E given by the Poisson equation:

$$\frac{dE}{dx} = 4\pi e (N_i - N_e) \qquad (1.3)$$

where N_e is the electron density, N_i is the ion density which is zero in this case, and x is the distance to one of the plates. Hence, the electrons give rise to the following potential difference ($E = -d\varphi/dx$):

$$\varphi = 2\pi e N_e d^2 = 2\pi j d^2 / v_e \qquad (1.4)$$

The output voltage is decreased by this value; the output voltage of the thermoemission converters amounts, typically, to about one volt. In (1.4) d is the distance between the plates, $j = eN_e v_e$ is the electric current density, and v_e is the electron current velocity.

From the above equations it readily may be estimated that the effect is absent for the practicable energy flux of about 1 W-cm^{-2} if the width of the gap between the plates is much less than 10 μm. This must be done for the plate temperature of about 2000 K, when there is intense evaporation of the material from the plate surface. Hence, the above condition is technologically unfeasible. However, this problem can be solved by filling the gap between the plates with plasma, which essentially will determine the parameters of the thermoemission converter.

The electrogasodynamic (EGD) generator is a less well-known device than the MHD generator. In the EGD generator, the gas flow containing ions of the same polarity (only negative or only positive) is directed through an electric field so that the ions are carried by the gas flow opposite to the field. Hence, the ions "produce" electric power by converting the energy of the gas flow. The output voltage of the EGD generator can be rather high, but its power and specific power are not high since the ion densities in the gas flow are small.

Interestingly, the concepts of the MHD and EGD generators and the thermoemission converter were suggested as early as the end of the last century. But the high-temperature materials necessary for constructing practicable systems have been developed only recently.

The same is true for the plasmatrons, the gas-discharge devices in which the electric energy is used for carrying out chemical reactions. Plasmatrons first were developed at the beginning of this century. However, because of the high cost of electric power at that time, it was too expensive to convert into chemical energy. Now plasmatrons are increasingly used in industrial applications, which make it possible to decrease considerably production areas, to obtain higher-quality products and to carry out processes in one stage, thus getting rid of the useless intermediate products. The above examples illustrate the fact that technological innovations are not necessarily due to the advances in pure science but can be initiated by developments in the technology itself.

Space plasma. Apart from the laboratory plasma, the attention of the scientists is increasingly drawn to the plasmas in the atmospheres of the earth and the planets, in the stars, including the sun, and in outer space.* Each of the above plasma types exists under rather special conditions. For instance, the plasma of the earth's atmosphere (hundreds of kilometers above the earth's surface) is created by the ultraviolet solar radiation. This plasma's parameters sharply vary according to certain processes occurring on the solar surface and to the parameters of the atmosphere itself. A few successful experiments have been carried out with temporary variation of the atmospheric plasma parameters in limited volumes of space.

The plasmas of stars differ greatly in their parameters. For instance, in the inner part of the sun where the thermonuclear fusion reaction occurs, the temperature is as high as 16 million degrees. The surface region of the sun about 1000 km thick which radiates most of the solar energy is referred to as the photosphere; the temperature of the photo-

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* Over 90% of the matter in the universe consists of charged particles, that is, it is in the plasma state.

sphere is about 6000 K and its distance from the sun's centre is 700 000 km. The region which is closer to the sun's centre is called the convective region since the energy is transported there with convective movement of the solar plasma in strong magnetic fields. Such movement of the solar plasma gives rise to the granular structure of the photosphere, development of the sun spots and other nonstationary phenomena on the sun's surface. However, the total solar power radiated in the optical range is fairly stable despite the nonstationary effects.

Over the sun's surface there is a low-density high-temperature region (the temperature of about 10^6 K) called the solar corona; it is a rather powerful source of ultraviolet radiation. The sun emits plasma from its surface. The stationary proton flux emitted by the solar corona is referred to as the solar wind. The plasma flow from the sun's surface gives rise to the interplanetary plasma. The electrons in this plasma are captured by the magnetic field of the earth and give rise to the radiation belts around the earth (at a distance of a few thousand kilometers). The high-energy electrons and protons produce various effects in the earth's atmosphere, in particular the auroras.

The interstellar plasma has a very low density and a temperature of about 3 K. The energy exchange between the particles of this plasma proceeds in a peculiar way via the interaction with the electromagnetic radiation field. The interstellar plasma is a source of information on the development of the universe.

2 Statistics of Weakly Ionized Gas

Distribution of particles by state. Let us assume that we consider an ensemble of a large number of particles and that each of the particles can be in one of the various states described by a set of quantum numbers *i*. We have to find how many particles of this system are in a given state. For instance, we consider a molecular gas and have to find the number of molecules in a given vibration-rotational state. Discussed below are problems of this type.

Let us consider a system of particles containing a definite number N of particles which does not vary with time. Let us denote the number of particles in the ith state by n_i; then the following relation must hold:

$$N = \sum_i n_i \tag{2.1}$$

Furthermore, our system of particles is closed, that is, it does not exchange energy with the outside world. Hence, if the total energy of the system is $\mathscr{E}$ and the energy of the particle in state i is $\mathscr{E}_i$, then the following relation is satisfied owing to conservation of the total energy of the system:

$$\mathscr{E} = \sum_i \mathscr{E}_i n_i \tag{2.2}$$

Our closed system is in the state which is termed thermodynamic equilibrium.

When the particles collide, they change their states, so that the number of particles in a given state is continuously changed. Hence, the probability that a given number of particles are in a given state is proportional to the number of possible realizations of this distribution.

Let $W(n_1, n_2, \ldots, n_i, \ldots)$ denote the probability that n_1 of the particles are in the first state, n_2 of the particles are in the second state, and so on, and let us calculate the number of possible realizations of this distribution. First, select from the total number N of the particles n_1 particles which are in the first state; there are $C_{n_1}^{N} = \frac{N!}{(N-n_1)!\,n_1!}$ ways to do that. Next, select n_2 particles which are in the second state from the remaining $N - n_1$ particles; this can be done in $C_{n_2}^{N-n_1}$ ways. Continuation of the procedure yields the following expression for the probability:

$$W(n_1, n_2, \ldots, n_i, \ldots) = \frac{AN!}{\prod_i n_i!} \tag{2.3}$$

where A is the normalization constant.

The Boltzmann distribution. Let us find the most probable number of particles, $\bar{n}_i$, in a given state i. It should be taken into account here that $\bar{n}_i \gg 1$, and for $n_i = \bar{n}_i$ the probability W of distribution of particles by state and the logarithm of this probability have maximums. Let us denote $dn_i = n_i - \bar{n}_i$ where $n_i \gg dn_i \gg 1$. Assuming that $n_i \gg 1$, we expand $\ln W$ at the maximum. Using the relation

$$\ln n_i! = \ln \prod_{m=1}^{n_i} m = \sum_{m=1}^{n_i} \ln m \approx \int_0^{n_i} \ln x \, dx$$

we find $d \ln n_i!/dn_i = \ln n_i$. From this relation and Eq. (2.3) we obtain

$$\begin{aligned} \ln W(n_1, n_2, \ldots, n_i, \ldots) \\ = \ln W(\bar{n}_1, \bar{n}_2, \ldots, \bar{n}_i, \ldots) \\ -\sum_i \ln \bar{n}_i \, dn_i - \frac{1}{2} \sum_i \frac{dn_i^2}{\bar{n}_i} \end{aligned} \qquad (2.4)$$

The maximum condition for this quantity gives

$$\sum_i \ln \bar{n}_i \, dn_i = 0 \qquad (2.5)$$

Making use of Eqs. (2.1) and (2.2), we find the following relations for dn_i:

$$\sum_i dn_i = 0 \qquad (2.6)$$

and

$$\sum_i \mathscr{E}_i dn_i = 0 \qquad (2.7)$$

The mean number of particles in a given state, $\bar{n}_i$, can be found from Eqs. (2.5)-(2.7). Multiply Eq. (2.6) by $-\ln C$ and Eq. (2.7) by $1/T$ where C and T are characteristic parameters of our system. Adding the resulting relations, we find that

$$\sum_i (\ln \bar{n}_i - \ln C + \mathscr{E}_i/T) \, dn_i = 0$$

Since this equation holds for any dn_i, the term in the parentheses is equal to zero. This equation yields the following expression for the most probable number of particles in a given state:

$$\bar{n}_i = C \exp(-\mathscr{E}_i/T) \qquad (2.8)$$

This distribution is termed the Boltzmann distribution.

Let us determine the physical meaning of the parameters C and T in Eq. (2.8). These parameters describe the particle system being considered and their values can be found from the additional conditions (2.1) and (2.2) which this system should meet. For instance, condition (2.1) yields $C \sum_i \exp(-\mathscr{E}_i/T) = N$. This shows that C is a normalization constant proportional to the total number of particles. The energy parameter T is termed the temperature of the system; according to Eq. (2.2) T can be related to the mean energy per particle.*

Before considering specific cases, we must make sure that for large $\bar{n}_i$ the probability that the number of particles in this state noticeably differs from $\bar{n}_i$ is low. Transform Eq. (2.4) taking into account Eq. (2.5):

$$W(n_1, n_2, \ldots, n_i, \ldots) = W(\bar{n}_1, \bar{n}_2, \ldots, \bar{n}_i, \ldots) \exp\left[-\sum_i \frac{(n_i-\bar{n}_i)^2}{2n_i}\right]$$

This shows that the probability is noticeably decreased if the difference between the number of particles and the mean value is $\Delta n_i \sim \bar{n}_i^{1/2}$. If the number of particles in the state is high, the relative variation $\Delta n_i/\bar{n}_i \sim \bar{n}_i^{-1/2}$ is small. Hence, the observed number of particles in this state practically coincides with the most probable number.

The statistical weight of a state and the distribution of particles in a gas. In the above discussion, the subscript i denoted one state of a particle. Now, let us take into account the fact that this state can be a degenerate one. Let us introduce the quantity g_i referred to as the statistical weight, which is equal to the number of states with the same quantum number. For instance, a rotational state of a molecule with the rotational quantum number J has the statistical weight of $2J+1$, that is, it equals the number of possible angular momentum projections on a given axis. Taking the sum over the degenerate states in Eq. (2.8), we can trans-

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* We express here the temperature in energy units and, hence, do not write the Boltzmann constant $k = 1.38 \times 10^{-16}$ erg-K^{-1} as is sometimes done.

form it into

$$\bar{n}_j = Cg_j \exp(-\mathscr{E}_j/T)$$

where the subscript j designates now a group of states. This equation yields a relation for particle densities:

$$N_j = N_0 \frac{g_j}{g_0} \exp\left(-\frac{\mathscr{E}_j}{T}\right) \qquad (2.9)$$

Here N_j and N_0 are the densities of particles in the jth and ground states, $\mathscr{E}_j$ is the excitation energy for the jth state, and g_j and g_0 are the statistical weights of the jth and ground states.

Let us find the statistical weight of the continuous spectrum states. The wave function of a free particle with momentum p_x moving along the axis x is given, up to an arbitrary factor, by $\exp(ip_x x/\hbar)$ if the particle moves in the positive direction and by $\exp(-ip_x x/\hbar)$ if the particle moves in the negative direction ($\hbar$ is Planck's constant h divided by 2π). Let us put the particle into a potential well with infinitely high walls so that the particle can move freely only in the region $0 \leqslant x \leqslant L$. Let us construct the wave function of the particle in the potential well as a combination of the above functions. The wave function of the particle must be zero at the walls of the well; the boundary condition for $x = 0$ shows that the wave function of the particle is proportional to $\sin(p_x x/\hbar)$, and the boundary condition for $x = L$ yields the possible values of the particle's momentum: $p_x L/\hbar = \pi n$ where n is an integer.

Hence, a particle with a momentum in the range from p_x to $p_x + dp_x$ can be in $dn = L\, dp_x/(2\pi\hbar)$ states if we take into account the sign of the momentum; if the particle is in the interval dx, the number of states for a free particle is

$$dn = \frac{dp_x\, dx}{2\pi\hbar} \qquad (2.10a)$$

The formula for the three-dimensional case is

$$dn = \frac{dp_x\, dx}{2\pi\hbar} \frac{dp_y\, dy}{2\pi\hbar} \frac{dp_z\, dz}{2\pi\hbar} = \frac{d\mathbf{p}\, d\mathbf{r}}{(2\pi\hbar)^3} \qquad (2.10b)$$

The number of states given by Eq. (2.10) is the statistical weight for the continuous spectrum states since it determines the number of states corresponding to a given range of continuously varying parameters. The quantity $d\mathbf{p}\, d\mathbf{r}$ is termed an element of phase space.

Now let us consider some particular Boltzmann distributions. First, let us study the distribution of diatomic molecules among the vibration-rotational states. For not too large vibrational quantum numbers v, the excitation energy of the vth vibrational level of the molecule is $\hbar\omega v$ where $\hbar\omega$ is the gap between the neighbouring vibrational levels in the energy space. Hence, according to Eq. (2.9) we find that

$$N_v = N_0 \exp(-\hbar\omega v/T) \qquad (2.11)$$

Since the total density of the molecules is $N = \sum_{v=0}^{\infty} N_v = N_0 \sum_v \exp(-\hbar\omega v/T) = N_0 [1 - \exp(-\hbar\omega/T)]^{-1}$, the density N_v of the molecules at the vth vibrational level is

$$N_v = N \exp\left(-\frac{\hbar\omega v}{T}\right)\left[1-\exp\left(-\frac{\hbar\omega}{T}\right)\right]^{-1} \qquad (2.12)$$

For the rotational state with the angular momentum J, the excitation energy is $BJ(J+1)$ where B is the rotational constant of the molecule. Since the statistical weight of a rotational state is $2J+1$, Eq. (2.9) yields the following expression for the density of molecules at a given vibration-rotational state:

$$N_{vJ} = N_v \frac{B}{T}(2J+1)\exp\left[-\frac{BJ(J+1)}{T}\right] \qquad (2.13)$$

Here we made use of the normalization condition $\sum_J N_{vJ} = N_v$ and assumed that $B \ll T$, which is typically the case.

Let us now consider the spatial distribution of particles in a uniform field. The particles are in a half-space; the force F acts upon each particle so that the potential energy U of each particle is $U = Fx$. Equation (2.9) yields the following distribution of the particles in space:

$$N(x) = N(0)\exp(-Fx/T)$$

where $N(0)$ is the particle density at the origin, and $N(x)$ is the particle density at the point x. A particular case of this formula is the distribution of the molecules in the earth's atmosphere by height under the effect of the gravitational field:

$$N = N(0)\exp(-Mgh/T) \qquad (2.14)$$

Here M is the molecule's mass, g is the acceleration of gravity, and h is the height above the earth's surface. For nitrogen Mg/T is 0.11 km^{-1} at room temperature, and so the atmospheric pressure varies noticeably when going up a few kilometers. Equation (2.14) is called the barometric height formula.

The Maxwell distribution. Let us consider one more distribution of particles by state, namely, the distribution over the velocities of gas particles. First, we shall discuss the one-dimensional problem. The number of the particles with the velocities in the range from v_x to $v_x + dv_x$ is designated as $n(v_x)\,dv_x$. The energy of these particles is $Mv_x^2/2$ (M is the particle mass) and the statistical weight is proportional to the number of states corresponding to the velocity range. The number of states is $dx\,dp_x/(2\pi\hbar)$ where $p_x = Mv_x$ is the particle's momentum, and dx is the coordinate range of the particle. The statistical weight in this case is seen to be proportional to the given velocity range and Eq. (2.8) yields

$$n(v_x)\,dv_x = C\exp\left(-\frac{Mv_x^2}{2T}\right)dv_x$$

where C is the normalization factor. The normalization condition $\int\limits_{-\infty}^{+\infty} n(v_x)\,dv_x = N$ (N is the total number of particles) yields $C = N\left(\frac{M}{2\pi T}\right)^{1/2}$. Let us introduce a new function $\varphi(v_x) = n(v_x)/N$ normalized to unity: $\int\limits_{-\infty}^{+\infty} \varphi(v_x)\,dv_x = 1$; hence, the probability that a particle has the velocity v_x is

$$\varphi(v_x) = \left(\frac{M}{2\pi T}\right)^{1/2}\exp\left(-\frac{Mv_x^2}{2T}\right) \qquad (2.15)$$

Equation (2.15) is termed the Maxwell distribution.

Write down the above result for the three-dimensional case. The number of particles having velocities in the range from $\mathbf{v}$ to $\mathbf{v} + d\mathbf{v}$ is $n(\mathbf{v})\,d\mathbf{v}$ where

$$\begin{aligned} n(\mathbf{v}) &= N\varphi(v_x)\,\varphi(v_y)\,\varphi(v_z) \\ &= N\left(\frac{M}{2\pi T}\right)^{3/2}\exp\left(-\frac{Mv^2}{2T}\right) \qquad (2.15a) \end{aligned}$$

where $v = (v_x^2 + v_y^2 + v_z^2)^{1/2}$ is the speed of a particle. Using Eq. (2.15a) we can determine the mean kinetic energy of a particle:

$$\overline{\frac{Mv^2}{2}} = \frac{\int\limits_0^\infty \frac{Mv^2}{2} \exp\left(-\frac{Mv^2}{2T}\right) v^2\,dv}{\int\limits_0^\infty \exp\left(-\frac{Mv^2}{2T}\right) v^2\,dv}$$

$$= -\frac{\partial}{\partial T^{-1}} \ln \int\limits_0^\infty \exp\left(-\frac{Mv^2}{2T}\right) v^2\,dv$$

$$= -\frac{\partial}{\partial T^{-1}} \ln aT^{3/2} = \frac{3}{2} T \qquad (2.16)$$

where a does not depend on the temperature. Hence, the mean kinetic energy of a gas particle is $3T/2$ and the mean kinetic energy per one degree of freedom is $T/2$. Equation (2.16) may be used for the definition of temperature.

The Saha distribution. Another case of interest we shall consider here is the equilibrium between continuous-spectrum and discrete-spectrum states. Let us find the relationship between the densities of electrons, ions, and atoms involved in the following processes:

$$A^+ + e \rightleftarrows A$$

where A^+ is the ion, e the electron, and A the atom. Let us assume the plasma to be quasineutral, that is, the ion density equals the electron density.

Equation (2.9) yields the following expression for the ratio between the mean number of the electrons, $\overline{n}_e = \overline{n}_i$, and the mean number of the atoms, $\overline{n}_a$, in the ground state:

$$\frac{\overline{n}_i}{\overline{n}_a} = \frac{g_e g_i}{g_a} \int \frac{d\mathbf{p}\,d\mathbf{r}}{(2\pi\hbar)^3} \exp\left(-\frac{I + p^2/2m}{T}\right)$$

Here g_e is the statistical weight of electrons, g_i and g_a are the statistical weights of the ion and the atom corresponding to their electron states, I is the ionization potential of the atom, p is the free electron momentum so that $I + p^2/2m$ is the energy needed for removing the electron from the atom and transferring to it the kinetic energy $p^2/2m$, and $d\mathbf{p}\, d\mathbf{r}/(2\pi\hbar)^3$ is the number of states in an element of phase space, that is, the states in the given range of coordinates and momentum of the particle.

Integration of this expression over the electron momentum yields

$$\frac{\bar{n}_i}{\bar{n}_a}=\frac{g_e g_i}{g_a}\left(\frac{mT}{2\pi\hbar^2)}\right)^{3/2}\exp\left(-\frac{I}{T}\right)\int d\mathbf{r}$$

Let the total volume of the system be V. When integrating this equation over volume, we should take into account that the state of the electron system is not changed if the coordinates of two electrons are interchanged. Therefore, to calculate the number of states per one electron, we must take into account only the volume per one electron. Hence we find $\int d\mathbf{r}=V/\bar{n}_e$. Using as notation for the electron density $N_e=\bar{n}_e/V$, the ion density $N_i=\bar{n}_i/V$, and the atom density $N_a=\bar{n}_a/V$, we can find the following relationship between these quantities:

$$\frac{N_e N_i}{N_a}=\frac{g_e g_i}{g_a}\left(\frac{mT}{2\pi\hbar^2}\right)^{3/2}\exp\left(-\frac{I}{T}\right) \tag{2.17}$$

This equation is called the Saha distribution.

Equation (2.17) can be written in the form of the Boltzmann distribution (2.9):

$$\frac{N_e'}{N_a}=\frac{g_{\text{cont.}}}{g_a}\exp\left(-\frac{I}{T}\right) \tag{2.18}$$

where $g_{\text{cont.}}=\frac{g_e g_i}{N_i}\left(\frac{mT}{2\pi\hbar^2}\right)^{3/2}$ is the effective statistical weight of the continuous spectrum. It can be readily seen that this weight is rather high for the ideal plasma. Owing to the high statistical weight of the continuous spectrum, the degree of plasma ionization is about unity for the temperatures $T\ll I$. These temperatures are low compared to the excitation energy of the atom. Hence, the relative number of excited atoms is small; at the temperature comparable to the excitation energy this is because almost all the atoms dissociate into ions and electrons.

Dissociation equilibrium in the molecular gas. Let us consider the equilibrium between atoms and molecules in the molecular gas where the following reaction occurs:

$$X + Y \rightleftarrows XY$$

The relationship between the densities of the atoms N_X and N_Y and the molecules N_{XY} which are in the ground vibration-rotational state is given by the Saha distribution (2.17):

$$\frac{N_X N_Y}{N_{XY}(v=0,\ J=0)} = \frac{g_X g_Y}{g_{XY}} \left(\frac{\mu T}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{D}{T}\right) \qquad (2.19)$$

Here μ is the reduced mass of the atoms X and Y, and D is the dissociation energy of the molecule. In contrast to the above case of ionization equilibrium where all the atoms were in the ground state, here most molecules are in excited states.

Making use of Eqs. (2.12) and (2.13), we can find the relationship between the total density of molecules N_{XY} and the density of molecules in the ground state N_{XY} ($v = 0$, $J = 0$):

$$N_{XY}(v=0, J=0) = \left[1 - \exp\left(-\frac{\hbar\omega}{T}\right)\right] \frac{B}{T} N_{XY}$$

Substituting this relation into Eq. (2.19), we obtain finally

$$\frac{N_X N_Y}{N_{XY}} = \frac{g_X g_Y}{g_{XY}} \left(\frac{\mu T}{2\pi\hbar^2}\right)^{3/2} \frac{B}{T} \times \left[1 - \exp\left(-\frac{\hbar\omega}{T}\right)\right] \exp\left(-\frac{D}{T}\right) \qquad (2.20)$$

Planck's distribution. Let us assume that radiation is in thermodynamic equilibrium with the walls of the vessel it fills and with the gas in the vessel. This radiation can be described by the temperature T equal to the temperature of the gas and the walls, and it is called black body radiation.

Let us find the mean number of photons in one state. The energy of a photon in a given state is $\hbar\omega$. Since photons obey Bose-Einstein statistics, any number of photons can be in a given state. From the Boltzmann formula (2.11) we find that the relative probability of n photons being in a given state is $\exp(-\hbar\omega n/T)$. The mean number of photons in the same state with a given energy is

$$\overline{n}_\omega = \frac{\sum\limits_n n \exp\left(-\frac{\hbar\omega n}{T}\right)}{\sum\limits_n \exp\left(-\frac{\hbar\omega n}{T}\right)} = \left[\exp\left(\frac{\hbar\omega}{T}\right) - 1\right]^{-1} \tag{2.21}$$

Equation (2.21) is referred to as Planck's distribution.

The laws of black body radiation. The energy of the electromagnetic radiation field per unit volume and unit frequency range is termed the spectral radiation density U_ω. Hence, the energy of the electromagnetic radiation field in the frequency range from ω to $\omega + d\omega$ filling volume V is given by $VU_\omega\, d\omega$. On the other hand, this energy can be written as $2\hbar\omega n_\omega V\, d\mathbf{k}/(2\pi)^3$ where factor 2 accounts for the two types of polarization of the transverse electromagnetic wave, $V\, d\mathbf{k}/(2\pi)^3$ is the number of states in the given volume of the phase space, n_ω is the number of photons in one state, and $\hbar\omega$ is the energy corresponding to this state. When we equate the above two expressions for the energy and make use of the dispersion relation $\omega = kc$ between the frequency ω and the wave vector $\mathbf{k}$ of the electromagnetic wave (c is the velocity of light), we find that the spectral radiation density is

$$U_\omega = \frac{\hbar\omega^3}{\pi^2 c^3} n_\omega \tag{2.22}$$

Replacing in Eq. (2.22) n_ω by Planck's distribution (2.21), we obtain

$$U_\omega = \frac{\hbar\omega^3}{\pi^2 c^3 \left(\exp\frac{\hbar\omega}{T} - 1\right)} \tag{2.23}$$

Equation (2.23) is called Planck's radiation formula. For the extreme case when $\hbar\omega/T \ll 1$, it yields the Rayleigh-Jeans formula

$$U_\omega = \frac{\omega^2 T}{\pi^2 c^3}, \quad \frac{\hbar\omega}{T} \ll 1 \tag{2.24}$$

For the other extreme case, $\hbar\omega/T \gg 1$, it yields the Wien formula

$$U_\omega = \frac{\hbar\omega^3}{\pi^2 c^3} \exp\left(-\frac{\hbar\omega}{T}\right), \quad \frac{\hbar\omega}{T} \gg 1 \tag{2.25}$$

Let us calculate the flux of radiation emitted by the surface of a black body, that is, the energy radiated from the unit surface area per unit time. Alternatively, this quantity may be interpreted as the radiation flux coming from a hole in a cavity with opaque walls filled with black body radiation. The black body surface radiates an isotropic flux $c\int_0^\infty U_\omega\, d\omega$ so that the energy flux $\frac{d\Omega}{4\pi}c\int_0^\infty U_\omega\, d\omega$ is emitted in the elementary solid angle $d\Omega$. Let us take the projections of the elementary radiation fluxes on the resultant flux vector which is normal to the emitting surface and take into consideration only the part of the flux which leaves the emitting body. Then we derive a formula for the resultant radiation flux:

$$J = \int_0^{\pi/2} \frac{c}{4\pi} \int_0^\infty U_\omega\, d\omega\, 2\pi \cos\theta\, d\cos\theta$$

$$= \frac{c}{4}\int_0^\infty U_\omega\, d\omega = \sigma T^4 \qquad (2.26)$$

where θ is the angle between the normal to the surface and the direction of the emitted photon, and the constant $\sigma =$ $= \frac{1}{4\pi^2 c^2 \hbar^3}\int_0^\infty \frac{x^3}{e^x - 1}\, dx = \frac{\pi^2}{60 c^2 \hbar^3} = 5.67 \times 10^{-12}$ W-cm^{-2}K^{-4}, is termed the Stefan-Boltzmann constant. Equation (2.26) represents the Stefan-Boltzmann law.

The radiation flux as a function of the parameters used in Eq. (2.26) can be determined by considering the dimensions involved. We deal with the following parameters: the characteristic energy of the photon T, Planck's constant $\hbar$ and the velocity of light c. The only combination of these parameters which has the dimension of the energy flux erg-cm^{-2}s^{-1} is $T^4/(\hbar^3 c^2)$, so that $J \sim T^4/(\hbar^3 c^2)$ as in Eq. (2.26).

3 The Ideal Plasma

The ideality of a plasma. We shall consider a plasma whose properties are similar to those of gas. When we observe a particle of such a plasma, the particle travels most of the time in a straight line at a constant velocity. To make this possible the energy of interaction between the particle and the surrounding particles at the mean distance between them must be considerably less than the mean kinetic energy of the particle. This condition is termed the condition of gaseousness for a system of particles, and the ideal plasma is the plasma which complies with this condition with respect to the interaction between charged particles.

Let us formulate the condition of ideality for a plasma. The interaction between two charged particles is described by the Coulomb potential whose absolute value is $|U(R)| = e^2/R$ where e is the charge of electron or singly charged ion, and R is the distance between the particles. The mean distance between the charged particles is of the order of $N_e^{-1/3}$ where N_e is the density of electrons equal to the density of ions. Hence, at the mean distance between the particles the energy of interaction between them is $|U| = e^2 N_e^{1/3}$. The mean kinetic energy of the charged particle is of the order of T where T is the temperature of the plasma expressed in energy units. Hence, we see that the parameters of an ideal plasma must satisfy the following condition:

$$\frac{N_e e^6}{T^3} \ll 1 \qquad (3.1)$$

In the discussion below we shall deal only with the ideal plasma. The nonideal plasma is not found in nature and so far it is impossible to create it under laboratory conditions.

Charged particles in a gas. We consider here the weakly ionized gas, that is, the gas in which the density of charged particles is considerably lower than the density of atoms or molecules. Nevertheless, many properties of the weakly ionized gas, in particular the electric ones, are due to the charged particles in it. For instance, the degree of ionization in the powerful discharge-driven molecular gas lasers is 10^{-7}-10^{-5}. The electric energy in these lasers is transferred from an external source to the electrons and then converted into the laser radiation energy. The presence of the electrons in the gas, though their concentration is low, determines the operation of the laser driven by electric discharge.

The interaction and collisions between charged and neutral particles determine many properties of weakly ionized gas. Some properties of weakly ionized gas are due only to the interaction between the charged particles. Though the concentration of charged particles in this case is low, the long-range Coulomb interaction between them can prove in some cases to be more significant than the short-range interaction between the charged and neutral particles. Below we shall discuss such properties of the weakly ionized gas which are due to the long-range interaction between the charged particles and which are not affected by the short-range interaction involving the neutral particles.

Screening of charge and field in a plasma. Let us consider penetration of the electric field produced by external charges into plasma. Since this field affects the distribution of charged particles in plasma, redistribution of charged particles affects in its turn the field. The result is that at a certain distance the external field is completely screened by the plasma.

Let us find the variation of potential of the external electric field in a plasma. Then we write down Poisson's equation:

$$\operatorname{div} \mathbf{E} = -\nabla^2\varphi = 4\pi e (N_i - N_e) \tag{3.2}$$

Here $\mathbf{E} = -\operatorname{grad} \varphi$ is the electric field strength, φ is the potential of the field, N_i is the density of the ions which are assumed to be singly charged, and N_e is the density of the electrons. Let us determine redistribution of the charged particles in the external field. The densities of the ions and electrons can be found from the Boltzmann distribution (2.9):

$$N_i = N_0 \exp(-e\varphi/T), \quad N_e = N_0 \exp(e\varphi/T) \tag{3.3}$$

Here N_0 is the mean density of the charged particles in the quasineutral plasma (that is, the plasma with the equal average densities of electrons and ions), and T is the temperature of the plasma.

Substitution of Eq. (3.3) into Poisson's equation (3.2) yields

$$\nabla^2\varphi = 8\pi N_0 e \sinh(e\varphi/T) \tag{3.4}$$

Assuming $e\varphi/T \ll 1$, we transform Eq. (3.4) into

$$\nabla^2\varphi = \varphi/r_D^2 \qquad (3.5)$$

where

$$r_D = \left(\frac{T}{8\pi N_0 e^2}\right)^{1/2} \qquad (3.6)$$

is the so-called Debye-Hückel radius.

Equation (3.5) has a solution which exponentially decreases far from the boundary. For instance, if a constant electric field penetrates a plasma through a plane boundary, Eq. (3.5) is transformed into $d^2\varphi/dx^2 = \varphi/r_D^2$ where the x axis is normal to the boundary plane. This equation yields the following expression for the electric field strength: $\mathbf{E} = \mathbf{E}_0 \exp(-x/r_D)$ where $\mathbf{E}_0$ is the electric field strength at the gas boundary, and x is the distance between a given point in the plasma and the gas boundary.

For the field of a test charge in a plasma, Eq. (3.5) has the following form:

$$\nabla^2\varphi \equiv \frac{1}{r}\frac{d^2}{dr^2}(r\varphi) = \frac{\varphi}{r_D^2}$$

where r is the distance from the test charge. If this test charge q is in the vacuum, the right-hand side of the equation, which is proportional to the density of charged particles in a plasma, vanishes and we obtain $\varphi = q/r$. The equation has the same solution for $r \to 0$, where there are no charged particles and no plasma effects. Hence, we derive the following expression for the potential of the test charge:

$$\varphi = \frac{q}{r}\exp\left(-\frac{r}{r_D}\right) \qquad (3.7)$$

Thus, the Debye-Hückel radius is the characteristic distance at which plasma screens the external field. The effect of the field of a charged particle on the surrounding particles is compensated at the same distance. Let us verify the validity of the condition $e\varphi/T \ll 1$ for the interaction of particles in a plasma. This condition was used to simplify Eqs. (3.4)-(3.5). Since the interaction between the charged plasma particles and the field of the test charge is manifested at distances of the order of the Debye-Hückel radius, we have to introduce into the above condition the potential at this distance. This, up to a numerical factor, yields the following form for the condition:

$$\frac{e^2}{r_D T} \sim \left(\frac{e^6 N_0}{T^3}\right)^{1/2} \ll 1$$

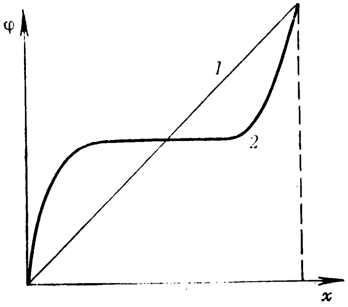


FIG. 2.
The potential drop in the gas gap as a function of coordinate x.
1—the density of charged particles is zero.
2—the gas contains charged particles and the Debye-Hückel radius is smaller than the gap width.

which proves to be the same as the condition for an ideal plasma (3.1).

Let us find the number of charged particles involved in screening the field of the test charge by a plasma. This number can be estimated as the number of charged particles in the sphere with the radius of the order of the Debye-Hückel radius; up to a numerical factor, it is given by

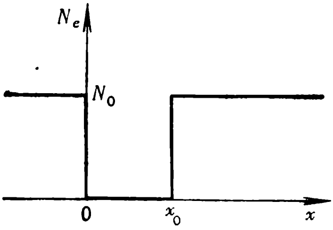
$$r_D^3 N_0 \sim \left(\frac{T^3}{e^6 N_0}\right)^{1/2} \gg 1$$

that is, this number is high for the ideal plasma.

Consider a gas gap in the external electric field. If the gas does not contain charged particles, the external field is uniform. If there are charged particles, the potential drops mainly at the edges at a distance of about the Debye-Hückel radius (Fig. 2). Let us evaluate the density of charged particles for uniform potential distribution in the gap 10 cm wide. For the characteristic electron temperature T of about 1 eV we find $N_e \ll 3 \times 10^4$ cm^{-3}. This is a very low density of electrons. For instance, in the glow gas discharge $N_e \sim 10^7$-10^{12} cm^{-3}.

Oscillations of plasma electrons. The characteristic size for plasma, as shown above, is the Debye-Hückel radius. Let us determine the characteristic time of plasma response to external fields. To do this, find what happens if all the electrons of plasma are removed a certain distance x_0 to the right starting from the plane $x = 0$ (Fig. 3). This will give rise to an electric field whose strength is given by Poisson's equation (3.2):

FIG. 3. Distribution of electrons for plasma oscillations.



$$\frac{dE}{dx} = 4\pi e\,(N_i - N_e)$$

If we assume that the electric field strength is zero for $x < 0$, for $x > x_0$ Poisson's equation yields $E = -4\pi e N_0 x_0$ where N_0 is the mean density of charged particles in the plasma. The movements of all the electrons in this field produce a change in the position of the gas boundary. The equation of motion for each of the electrons can be written as

$$m\frac{d^2(x + x_0)}{dt^2} = eE$$

where m is the electron mass, and x is the distance of the electron to the boundary x_0; the distance is not related to the phenomena being considered and does not depend on time. Thus, the equation of motion for the electron may be written as

$$\frac{d^2x_0}{dt^2} = -\omega_0^2 x_0 \qquad (3.8)$$

where

$$\omega_0 = \left(\frac{4\pi N_0 e^2}{m}\right)^{1/2} \qquad (3.9)$$

is the Langmuir, or plasma, frequency.

The solution of Eq. (3.8) shows that in this case the motion of the electrons has an oscillatory character with a frequency ω_0. Hence, $1/\omega_0$ is the characteristic time of plasma response to an external effect (for instance, if the external field is applied instantly, the distribution of the field in the plasma shown in Fig. 3 is established in time of about $1/\omega_0$). Note that $r_D\omega_0 = (2T/m)^{1/2}$ is the thermal velocity of electrons. Hence, the characteristic time of plasma response to external effects is the time during which the electrons cover the distance of about the Debye-Hückel radius.

The skin effect. Let us consider penetration of slow-varying fields into the plasma. The characteristic frequency ω of variation of these fields is small compared to the plasma frequency; hence, we can apply Ohm's law for the plasma, $\mathbf{j} = \sigma\mathbf{E}$ where $\mathbf{j}$ is the current density in plasma, $\mathbf{E}$ is the electric field strength, and σ is the plasma conductivity corresponding to the constant electric field. To describe the variation of the fields, we must add to Ohm's law the Maxwell equations

$$\left.\begin{aligned} \operatorname{curl}\mathbf{H} &= \frac{4\pi}{c}\mathbf{j} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \\ \operatorname{curl}\mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \quad \operatorname{div}\mathbf{H} = 0 \end{aligned}\right\} \tag{3.10}$$

where $\mathbf{H}$ is the magnetic field strength.

Let us assume that the characteristic frequency ω of variation of the external fields is small compared to the plasma conductivity σ so that, taking into account Ohm's law, the first Maxwell equation can be transformed into curl $\mathbf{H} = 4\pi\sigma\mathbf{E}/c$. Substituting the resulting expression for the electric field strength into the second Maxwell equation (here we make use of the relation curl curl $\mathbf{a}$ = grad div $\mathbf{a} - \nabla^2\mathbf{a}$) and taking into account the third Maxwell equation, we derive an equation for the magnetic field:

$$\frac{\partial \mathbf{H}}{\partial t} = \frac{c^2}{4\pi\sigma}\nabla^2\mathbf{H} \tag{3.11}$$

A similar equation can be derived for the electric field.

Using Eq. (3.11) and dimensional analysis, we find that the characteristic size corresponding to the distribution of fields is

$$l \sim \left(\frac{c^2}{4\pi\omega\sigma} \right)^{1/2} \tag{3.12}$$

If this size is small compared to the size of the plasma, the external fields and the currents in plasma are concentrated only at the surface of the plasma and penetrate into it to a depth of about l. This phenomenon is referred to as the skin effect, and the layer at the plasma surface where the external fields penetrate and where the plasma currents flow is termed the skin layer. Equation (3.12) shows that the smaller the skin-layer thickness is, the higher is the plasma conductivity and the frequency of variation of the fields.

Let us make some numerical estimates. The conductivity of the plasma of the upper atmosphere at the height of about 100 km is about 10^9 s^{-1} and the plasma frequency is about 3×10^7 s^{-1}. For the frequencies of the order of the plasma frequency the electromagnetic waves penetrate into plasma for fractions of a meter, that is, the depth of penetration is considerably less than the depth of the atmosphere. It can be seen that the electromagnetic signals whose frequencies are lower than the plasma frequency cannot pass through the upper atmosphere.

4 Elementary Processes in Plasma

Act of collision of particles in a plasma. Let us analyze the collision of two particles in plasma. We have to find a parameter for describing this act of collision. Denote the first particle by A and the second particle by B. Let us consider the collision which alters the inner state of the particle A; describe the state of the particle A by the subscript i before the collision and by the subscript f after the collision. Assume that each collision of the test particle A with particle B can result only in transition of particle A into state f. The probability $W(t)$ that particle A has not altered its state by time t is given by the following equation:

$$\frac{dW}{dt} = -\nu_{if} W \tag{4.1}$$

where ν_{if} is the frequency of transition of particle A from state i to state f following the collision with particle B.

Let us introduce a frame of reference where the test particle A is at rest. The higher is the frequency of variation of the particle's state, the larger the incident flux j of particles B. The quantity ν_{if}/j does not depend on the density of the B particles. Hence, this ratio characterizes the act of collision between two particles. It is termed the cross section of inelastic collision of particles. This cross section can be defined as the ratio between the probability of transition from one state of a particle to another state per unit time and the incident flux of particles.

If all the B particles travel at the same velocity, the flux of the B particles in the frame of reference linked to particle A is $|\mathbf{v}_A - \mathbf{v}_B|\,[B]$. Here $\mathbf{v}_A$ and $\mathbf{v}_B$ are the velocities of the particles A and B, and $[B]$ is the density of the B particles. Hence, we can write the following relationship between the frequency ν_{if} of transitions between the states and the cross section σ_{if} of transition:

$$\nu_{if} = [B]\,|\mathbf{v}_A - \mathbf{v}_B|\,\sigma_{if} \qquad (4.2)$$

where the cross section σ_{if} of transition depends only on the relative velocity of collision of the particles.

When particles A and B have different velocities described by a certain distribution, the frequency of transitions of particle A following collisions with particles B is

$$\nu_{if} = [B]\,\langle |\mathbf{v}_A - \mathbf{v}_B|\,\sigma_{if}\rangle = [B]\,\langle k_{if}\rangle \qquad (4.3)$$

The angle brackets here denote averaging over the relative velocities of the particles, and $k_{if} = |\mathbf{v}_A - \mathbf{v}_B|\,\sigma_{if}$ is the rate constant of the process. This rate constant also characterizes the act of collision between the particles. The rate constant of the transition process is useful when measuring or analyzing the frequency of transitions between the states in a gas or a plasma, that is, when we are interested in the frequency of transitions averaged over the velocities of the particles.

Using the rate constants for the process of transition, let us derive the balance equation for the A particles which are in a given state i. The balance equation describes variation of the particle due to appearance of new particles in the given state and transitions of the particles from this state into other states. The form of the balance equation depends on the processes resulting in the transitions involving the given state. Denote the density of A particles in the state f by N_f and assume that all the transitions between the states of particle A are due to collisions with B particles. Using the definitions of the rate constant for the transition process, we can write the balance equation for particles A in the following form:

$$\frac{dN_i}{dt}=[B]\sum_f k_{fi}N_f-[B]N_i\sum_f k_{if} \tag{4.4}$$

Here N_i is the density of A particles in state i and k_{if} is the rate constant for the process of transition of particle A from state i to state f. The balance equation (4.4) can be readily extended to cover other processes.

Elastic collision of particles. Let us consider elastic collision of two particles, that is, the collision which does not alter the internal states of the particle and changes only their directions of motion. The motion of these particles is described by the following equations (Newton's equations):

$$M_1\ddot{\mathbf{R}}_1=-\frac{\partial U}{\partial \mathbf{R}_1},\quad M_2\ddot{\mathbf{R}}_2=-\frac{\partial U}{\partial \mathbf{R}_2}$$

Here $\mathbf{R}_1$ and $\mathbf{R}_2$ are the radius vectors of the respective particles, M_1 and M_2 are their masses, U is the potential of the interaction between the particles, which depends on the distance between them: $U \equiv U(\mathbf{R}_1-\mathbf{R}_2)$, $\partial U/\partial \mathbf{R}_i$ is the force with which the other particle acts on a given particle, and $\partial U/\partial \mathbf{R}_1=-\partial U/\partial \mathbf{R}_2$.

Let us introduce new radius vectors: the vector of the centre of mass of the particles, $\mathbf{R}_c=(M_1\mathbf{R}_1+M_2\mathbf{R}_2)/(M_1+M_2)$, and the separation between them, $\mathbf{R}=\mathbf{R}_1-\mathbf{R}_2$. The Newton equations in these variables have the following form:

$$(M_1+M_2)\ddot{\mathbf{R}}_c=0,\quad \mu\ddot{\mathbf{R}}=-\frac{\partial U}{\partial \mathbf{R}}$$

Here $\mu=M_1M_2/(M_1+M_2)$ is the reduced mass of the particles. It is seen that the centre of mass travels at a constant velocity, so that to find the cross section of elastic collision we have to analyze the motion of one particle with a mass μ in a central force field. Though the above discussion was in terms of classical mechanics, in quantum mechanics the situation is the same. Indeed, in quantum mechanics there is also the free motion of the centre of mass in the absence of external forces, and collision depends only on the distance between the particles *R*.

Figure 4 shows the path of a particle with mass μ in the centre-of-mass frame of reference when the central potential of interaction between the colliding particles depends only on the distance $|\mathbf{R}_1 - \mathbf{R}_2|$ between the particles. The parameters describing the collision also are shown in Fig. 4. Let us find the relationship between the impact parameter ρ and the distance of closest approach r_0. To do this, make use of the conservation of the momentum, which is $\mu v \rho$ for large distances between the particles and $\mu v_\tau r_0$ at the distance of closest approach; here $v = |\mathbf{v}_1 - \mathbf{v}_2|$ is the relative velocity of the particles, and v_τ is the tangential component of the velocity at the distance of closest approach where the normal component of the velocity is zero. Conservation of energy gives $\mu v_\tau^2/2 = \mu v^2/2 - U(r_0)$ and we find the following relationship between the impact parameter and the distance of closest approach of the particles:

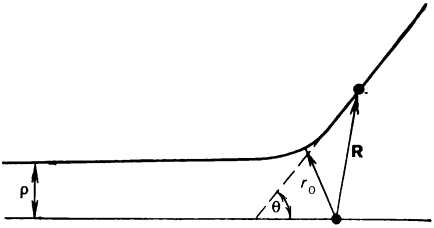


FIG. 4.
The trajectory of a particle in a central field in the centre-of-mass frame of reference: ρ is the impact parameter, **R** is the radius vector of the particle with the reduced mass, θ is the scattering angle in the centre-of-mass frame of reference, and r_0 is the distance of closest approach, that is, the minimum distance between the particles for a given impact parameter.

$$1 - \frac{\rho^2}{r_0^2} = \frac{U(r_0)}{\varepsilon} \qquad (4.5)$$

where $\varepsilon = \mu v^2/2$ is the energy of the particles in the centre-of-mass frame of reference.

To determine the cross section of scattering of the particles, we use the centre-of-mass frame of reference where scattering can be considered as motion of one particle in a central force field. Let us define the differential cross section of elastic scattering as the ratio of the number of collisions per unit time that scatter the particles into the elementary solid angle $d\Omega$ at a definite angle to the flux of incident particles. Consider a beam of particles of density N and velocity v falling on the scattering centre so that the flux of particles is Nv. In a central force field the particles scattered into the elementary solid angle $d\Omega = 2\pi d(\cos\theta)$ at the angle θ have an impact parameter from ρ to $\rho + d\rho$ since the scattering angle θ depends on the impact parameter. The number of particles scattered per unit time into the given elementary solid angle is $2\pi\rho\, d\rho\, Nv$, so that by definition the differential cross section in the central force field is

$$d\sigma = 2\pi\rho\, d\rho \qquad (4.6)$$

Elastic scattering of particles gives rise to many macroscopic parameters of gas and plasma: these parameters depend on the cross sections of elastic scattering of the gas and plasma particles averaged over the scattering angles. Of importance here is the average cross section, the main contribution to which is given by scattering at large angles. An estimate of this cross section can be made from Eq. (4.6).

Let us estimate the cross section of scattering at large angles. For large-angle scattering the interaction potential at the distance of closest approach is comparable to the kinetic energy of the particles; hence, the cross section for large-angle scattering is given by

$$\sigma = \pi\rho_0^2 \qquad (4.7)$$

while

$$\frac{U(\rho_0)}{\varepsilon} \sim 1$$

The most often used averaged cross section of elastic scattering is the so-called diffusion, or transport, cross section, which is defined as

$$\sigma^* = \int (1 - \cos\theta)\, d\sigma \qquad (4.8)$$

where θ is the scattering angle. The small scattering angles do not contribute to the diffusion cross section since they appear in the integrand with a weight factor $\theta^2/2$. All the macroscopic parameters related to the elastic scattering of electrons by atoms can be expressed through the diffusion cross section. Some transport parameters due to collisions of atoms and molecules can be expressed through another averaged cross section, $\sigma^{(2)} = \int (1 - \cos^2\theta)\, d\sigma$. It may be seen that this averaged cross section also depends on the large-angle scattering.

Another widely used parameter is the so-called gas-kinetic cross section σ_g, which is defined as the averaged cross section of elastic scattering of gas atoms or molecules at large angles for thermal energies. This rough parameter amounts to about 10^{-15} cm^2 (see Appendix 1).

Since $\nu \sim Nv\sigma$ is the frequency of collisions between the test particle and the gas particles, the quantity $\tau \sim 1/\nu$ is the characteristic time between two successive collisions for the given test particle, and the quantity $\lambda = v\tau \sim \sim (N\sigma)^{-1}$ is the distance covered by the test particle between two successive collisions referred to as the mean free path. As follows from the definition of the mean free path, its value can be only estimated similar to the characteristic cross section for particle collision.

The total cross section of scattering and the cross section of capture. Let us consider the total cross section of the elastic scattering of particles, $\sigma_t = \int d\sigma$. In classical terms the total cross section must be infinite. Indeed, the classical particles are scattered at any distance from each other and at any energy of interaction, and the total cross section must take into account all these scatterings. Hence, the total cross section of scattering is essentially a quantum quantity and depends on $\hbar$.

Let us estimate the total cross section of particles governed by laws of classical mechanics. The variation of the particle's momentum following a collision with another particle is given by

$$\Delta \mathbf{p} = \int_{-\infty}^{+\infty} \mathbf{F}\, dt$$

where $\mathbf{F} = -\partial U/\partial \mathbf{R}$ is the force with which one particle acts upon the other particle, and U is the potential of the interaction between the particles. Hence, we find that $\Delta p \sim U(\rho)/v$ where ρ is the impact parameter. According to the Heisenberg uncertainty principle, the value of Δp can be determined up to an accuracy of $\hbar/\rho$. Hence, the main contribution to the total cross section of scattering is given by the impact parameters that satisfy the relation $\Delta p(\rho) \sim \hbar/\rho$, and the total cross section estimate is given by

$$\sigma_t \sim \rho_t^2 \tag{4.9}$$

while $\rho_t U(\rho_t)/(\hbar v) \sim 1$.

In particular, if $U(R) = CR^{-n}$, the total cross section is given by

$$\sigma_t \sim (C/\hbar v)^{2/(n-1)} \tag{4.10}$$

Since the value of the total cross section is determined by a quantum effect, it tends to infinity for the classical limit. This can be demonstrated with the above equations by tending Planck's constant to zero.

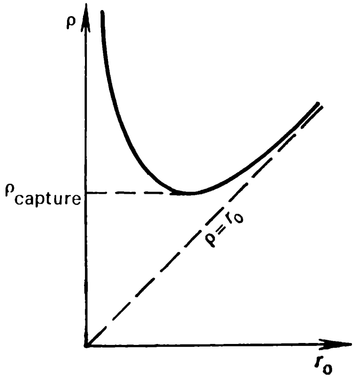
Let us consider the scattering of particles in the case of attractive interaction with the potential increasing at small distances faster than $1/R^2$. Assuming that the variation of the interaction potential is monotonic, we can use Eq. (4.5) to derive the function $\rho(r_0)$ shown in Fig. 5. We see that no distance of closest approach exists for impact parameters ρ less than ρ_{capture}. Such collisions result in particles approaching each other to infinitesimal distances, or one particle is said to capture the other one. For the inverse-power potential $U(R) = -CR^{-n}$ the capture cross section is

$$\sigma_{\text{capture}} = \pi\rho^2_{\text{capture}} = \frac{\pi n}{n-2}\left[\frac{C(n-2)}{2\varepsilon}\right]^{2/n} \tag{4.11}$$

Here the relationship between the cross section and the parameters is the same as in Eq. (4.7). For instance, for the polarization interaction between an ion and an atom $U(R) = -\beta e^2/(2R^4)$ where β is the polarizability of the atom and e is the charge of the electron, the cross section for the polarization capture of the ion by the atom is

$$\sigma_{\text{capture}} = 2\pi \left(\frac{\beta e^2}{\mu v^2}\right)^{1/2} \tag{4.12}$$

FIG. 5.
The impact parameter ρ as a function of the distance of closest approach r_0 for an attractive potential which increases faster than $1/R^2$ at small distances.



The condition of gaseousness and the ideality of plasma. The condition of gaseousness for a system of particles can be formulated in terms of the cross section of particle collision. We shall define a gas as a system of weakly interacting particles in which every particle most of the time behaves like a free particle, that is, travels in a straight line at a constant velocity and only for a relatively short time interacts strongly with other particles, thus changing its velocity, direction of motion and, possibly, the internal state. The strong interaction between the particles will be characterized by the cross section of elastic scattering at large angles. Then a test particle interacts strongly with another gas particle if the second particle is in the sphere of the volume of about $\sigma^{3/2}$ around the test particle. The gas volume per particle is $1/N$ where N is the density of particles. This leads to the following form of the condition of gaseousness:

$$N\sigma^{3/2} \ll 1 \tag{4.13}$$

Under this condition the average potential of interaction between the test particle and the surrounding particles is much lower than the particle's mean energy.

Let us apply the above condition to a system of charged particles. Since the interaction between charged particles is governed by the Coulomb law, $|U(R)| = e^2/R$,. Eq. (4.7) yields the following expression for the cross section of elastic collision between charged particles:

$$\sigma \sim e^4/T^2 \qquad (4.14)$$

where T is the mean energy of the charged particles, that is, their temperature expressed in energy units. Note that Eq. (4.13), which is a condition of the ideality of plasma, coincides with Eq. (3.1):

$$N_e e^6/T^3 \ll 1 \qquad (4.15)$$

The types of elementary processes. Given below is a general description of the elementary processes occurring in a weakly ionized plasma, of the mechanisms of these processes, and of the magnitudes of their cross sections. The facts are presented in Tables 1-3. More detailed information on some of the processes will be presented later. Here we shall discuss some common features of the mechanisms of certain processes.

Some processes of collision between electrons and molecules occur via the stage of formation of intermediate bound states of the electron and the molecule. For instance, Fig. 6 shows the terms of the diatomic molecule and the negative molecular ion. The molecular terms are the energy levels of the molecule depending on the distance between the nuclei. The negative molecular ion is in the auto-ionization state, that is, it has a finite lifetime and can decay into a molecule and a free electron. Therefore, the energy level of a negative ion cannot be determined precisely and possesses a certain width.

Taking into account the auto-ionization state of the negative molecular ion, let us describe the process of collision between the electron and molecule which occurs via formation of this state. Assume that at the moment of collision between an electron and a molecule, the distance between the nuclei is R_1. Since the characteristic time of the electronic

Table 1 The Elementary Processes of Collision Between Electrons and Atoms or Molecules

Elementary process	Process scheme	Cross section or constants of process, the process mechanism
Elastic collision between an electron and an atom or a molecule	$e+A \rightleftarrows e+A$	Typically, the cross section is of the order of the gas-kinetic cross section
Inelastic transitions between electron states of an atom or molecule	$e+A \rightarrow e+A^*$	(the asterisk denotes an excited electron state)
Ionization of an atom or molecule by electron impact	$e+A \rightarrow 2e+A^+$	See Sec. 5
Transitions between rotational levels of a molecule	$e+AB(J) \rightarrow e+AB(J')$ (J is the angular momentum of the molecule)	The cross section is, typically, less by one or two orders of magnitude than the cross section of a collision between a slow electron and a molecule. A selection rule determines the difference $J-J'$ according to the molecule species and the energy of collision
Transitions between the vibrational levels of a molecule	$e+AB(v) \rightarrow e+AB(v')$ (v is the vibrational quantum number)	Processes proceed via the formation of the auto-ionization state AB^- of molecule AB. For a given energy process, the probability depends on the probability of capturing the electron of the given energy at the auto-ionization level. The cross section has a resonance energy dependence, and the maximum cross section is less or of the order of the atomic cross section
Dissociative attachment of an electron to a molecule	$e+AB \rightarrow A^-+B$	
Dissociative recombination	$e+AB^+ \rightarrow A+B^*$	

Dissociation of a molecule by collision with an electron	$e+AB\rightarrow e+A+B$	Excitation of a molecule onto the repulsion term. The cross section dependence is the same as for the excitation of molecular levels by collision with an electron
Attachment of an electron to a molecule in three-particle collisions	$e+A+B\rightarrow A^{-}+B$	See Sec. 5
Recombination of an electron and an ion in three-particle collisions	$e+A^{+}+C\rightarrow A^{*}+C$	

Table 2 The Elementary Collision Processes for Atoms and Molecules

Elementary process	Process scheme	Cross section or constants of process, the process mechanism
Elastic collision of atoms and molecules	$A+B \rightleftarrows A+B$	For thermal energies of collision, the cross section has the gas-kinetic value (10^{-15} cm^2) and varies by an order of magnitude
Excitation of electron levels in collisions of atoms	$A+B \rightarrow A+B^*$	Adiabatically, these processes have a small probability for energies of a few eV and the respective cross sections are smaller than atomic cross sections by a few orders of magnitude; the asterisk denotes an excited electron state
Ionization with collisions of atoms	$A+B \rightarrow A+B^+ + e$	
Transitions between vibrational levels of molecules	$A+BC(v) \rightarrow A+BC(v')$	For thermal energies, the cross section is smaller than a gas-kinetic cross section by several orders of magnitude and strongly depends on the nature of the molecules
Transitions between rotational levels of molecules	$A+BC(J) \rightarrow A+BC(J')$	For thermal energies, the cross section of collision is of the order of atomic cross section
De-excitation of excited electron state in collision with molecule	$A^*+BC \begin{array}{l} \nearrow A+BC \\ \searrow A+B+C \end{array}$	In the first channel the molecule is in an excited vibrational state. Cross section of collision amounts to a gas-kinetic cross section. De-excitation by collision with an atom has a low probability
Associative ionization	$A+B^* \rightarrow AB^+ + e$	For a small number of excited states, the cross section is a gas-kinetic one; for other states it is small

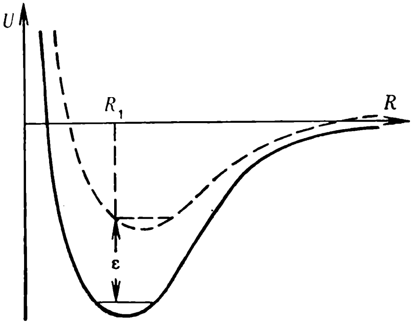
Penning effect	$A^* + B \rightarrow A + B^+ + e$	Atom A^* is in a metastable state, its excitation energy is higher than the ionization potential of atom B. On the average, the cross section of the process is of the order of a gas-kinetic one
Transfer of excitation	$A^* + B \rightarrow A + B^*$	
Spin exchange and the variation of hyperfine-structure states	$A(\downarrow) + B(\uparrow) \rightarrow$ $\rightarrow A(\uparrow) + B(\downarrow)$	(arrows denote spin orientation of the electrons)
Transitions between fine-structure states	$A + B(j) \rightarrow A + B(j')$	(j denotes the total angular momentum of atom) For a small variation of transition energy, these resonance processes can have cross sections larger than the gas-kinetic ones
Depolarization of an atom	$A + B(\uparrow) \rightarrow A + B(\downarrow)$	(arrows denote orbital angular momentum orientation of the electrons)
Chemical reactions	$A + BC \rightarrow AB + C$	
Production of molecules in three-particle collisions	$A + B + C \rightarrow AB + C$	For thermal energies, the rate constant of a three-particle process is, typically, of the order of 10^{-32} cm^{6}s^{-1} (see Sec. 5)

Table 3 The Elementary Collision Processes Involving Ions

Elementary process	Process scheme	Cross sections or constants of process, the process mechanism
Resonance charge exchange	$A^+ + A \to A + A^+$	The cross section of the process is larger than the gas-kinetic cross section (see Appendix 1)
Nonresonance charge exchange	$A^+ + B \to A + B^+$	When condition (4.16) is not satisfied, the cross section is of the same order as that of the resonance charge exchange
Mutual neutralization of ions	$A^+ + B^- \to A^* + B$	The cross section of the process is, typically, larger than the atomic cross section (see Sec. 5)
Decay of a negative ion in collision with an atom	$\begin{array}{l} A + B^- \to AB + e \\ \downarrow \\ A + B + e \end{array}$	Depending on the process mechanism (see the text), the process has a threshold determined by the behaviour of terms. This threshold can be considerably larger than the binding energy of the electron in a negative ion
Molecular ion reaction	$A + BC^+ \begin{array}{l} \nearrow AB^+ + C \\ \searrow AB + C^+ \end{array}$	These processes are similar to chemical reactions
Transformation of atomic ions into molecular ions in three-particle collisions	$A^+ + B + C \to AB^+ + C$	For thermal energies, the rate constant of the process is of the order of 10^{-31} cm^{6}s^{-1} for simple systems, and for complex molecules the rate constant can be considerably larger (up to 10^{-27} cm^{6}s^{-1})
Ion-ion recombination in three-particle collisions	$A^+ + B^- + C \to A + B + C$	See Sec. 5

processes is considerably less than the characteristic time of the nuclear processes, the distance between the nuclei in the molecule will not be changed during this process of electronic transition (the Franck-Condon principle). Therefore, the molecule can capture the electron if the energy of the incident electron is close to the energy difference between the negative ion and the molecule for a given distance between the nuclei (see Fig. 6). Since this energy difference varies with the distance between the nuclei, the range of the electron energies at which it can be captured by the molecule producing a negative ion is extended to a width of a fraction of an electron volt. The cross section of the electron capture by the molecule exhibits a resonance with a width of a fraction of an electron volt.

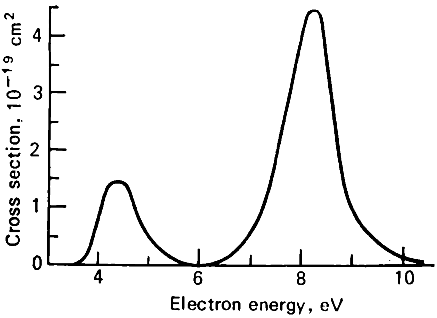
Fig. 6. The terms of a molecule (solid line) and a negative ion (dashed line). ε is the energy of the captured electron when the distance between the nuclei is R_1.



A variety of processes of electron scattering on atoms or molecules depend on the formation of the bound state of the electron attached to these particles. For elastic and inelastic collisions between the electron and the atom this gives rise to resonances on the cross section of scattering versus the collision energy curve. Some processes proceed only via the auto-ionization state of the electron and the atom or molecule. For such processes the cross section versus the electron energy curve exhibits one or a group of resonances whose position and character depend on the properties of the auto-ionization state. Among such processes are the excitation of the vibrational levels of the molecule, dissociative attachment of the electron to the molecule, and dissociative recombination of the electron and the molecular ion. An illustration of the processes of this type is presented in Fig. 7, which shows the cross section of the dissociative attachment of the electron to the carbon dioxide molecule as a function of the electron energy.

Inelastic collisions of atomic particles. The term atomic particle refers here to an atom, ion, or molecule. Let us consider inelastic collision of atomic particles which results in a change in the internal state of one of them, for instance, in its electronic state. If the characteristic times of the particle approach are high compared to the characteristic electronic times, then the electrons have time "to follow" the motion of the nuclei and for each nuclear configuration to establish the same distribution as for the infinitely slow motion of the nuclei. It is clear that for infinitely slow collisions the system finally will return back to the initial state and no transitions will occur. Therefore, if the time of approach of the nuclei is small compared to the characteristic electronic times, the probability of inelastic collision with electronic transition is small.

FIG. 7.
The cross section of dissociative attachment of an electron to a carbon dioxide molecule ($e+CO_2 \longrightarrow O^- + CO$) as a function of the energy of incident electron. The maxima at the electron energies 4.4 and 8.2 eV correspond to the positions of the auto-ionization levels of the negative ion CO_2^-.



The characteristic time of approach (or collision) of the nuclei is of the order of a/v where v is the relative velocity of the particles, and a is of the order of atomic size. The characteristic electronic time for the given transition is of the order of $\hbar/\Delta\varepsilon$ where $\Delta\varepsilon$ is the energy difference of the levels between which the transition occurs. Hence, the suggested criterion can be written as follows:

$$\frac{\Delta\varepsilon a}{\hbar v} \gg 1 \qquad (4.16)$$

The criterion (4.16) is called the adiabatic Massey principle; if this inequality is satisfied, the transition proves to be adiabatically unlikely.

There is also low probability, adiabatically, of such events as the transitions between the electronic states of atoms following collisions with atoms, ionization in the atom-atom or atom-molecule collisions at the energies of the order of a few electron volts, and the transitions between the vibrational molecular levels at thermal energies. The cross sections of these processes are smaller than the gas-kinetic cross sections by a few orders of magnitude. For other processes, the adiabatic criterion (4.16) is satisfied only at large distances between the nuclei. The interaction between the atomic particles draws together the electronic levels between which the transition occurs. Hence, the adiabatic criterion (4.16) is not satisfied starting from a certain distance between the particles and the process proceeds with a high probability.

This can be illustrated by the process of decomposition of a negative ion following the collision with the atom: $A^- + B \rightarrow AB + e \rightarrow A + B + e$. At a certain distance between the nuclei the term of the A^-B system intersects the boundary of the continuous spectrum, that is, the AB term. Therefore, there is a higher probability for the electron to be detached from the negative ion.

Another process of this type is the associative ionization when an atom and an excited atom collide: $A + B^* \rightarrow$ $\rightarrow AB^+ + e$. Figure 8 presents the terms for the initial and final channels of the reaction. Similar to the above processes, the mechanism of this reaction is determined by the initial-state term moving into the continuous-spectrum region. The reaction results in decomposition of the autoionization state AB^* giving rise to a free electron.

Some processes occur with a small variation in the electronic energy; they are termed the resonance processes. These processes satisfy a condition opposite to criterion (4.16) even for large distances between the nuclei. Hence, the resonance processes are effective when the colliding particles pass at large distances from each other, the distances considerably greater than the sizes of the particles. Hence, the resonance processes can have cross sections which are considerably larger than the areas of the particles cross sections.

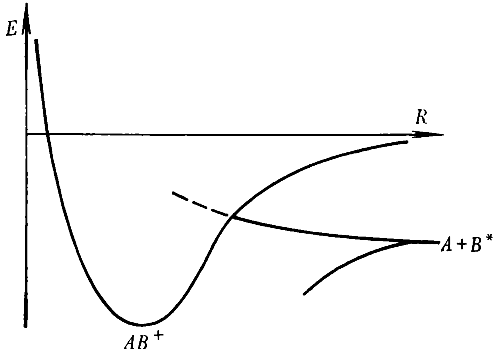


FIG. 8. Atomic terms in associative ionization and dissociative recombination: $A + B^* \rightleftarrows AB^+ + e$.

The resonance processes include the resonance charge exchange, the processes resulting in depolarization of the atom or in variation of its fine or hyperfine electronic structure, mutual neutralization from collision between ions, transfer of excitation, etc.

Charge exchange and similar processes. Let us consider the processes in which the electron is transferred from the field of one atomic core to the field of another atomic core. Among these processes are the resonance charge exchange, mutual neutralization of negative and positive ions, spin exchange, and some types of excitation transfer accompanied by exchange of one or two valence electrons. For instance, Fig. 9 represents the potential of the interaction between the electron and the atomic cores in the case of the resonance charge exchange for the cores of the same type.

Let us estimate the cross section of the resonance charge exchange as a function of collision velocity. If the distance between the nuclei of the atomic cores remains constant (Fig. 9), the transfer of the electron from one atomic core to another will occur with a certain frequency ω. The main contribution to the cross section of the charge exchange is given by those collisions for which the characteristic time of particles collision a/v is comparable to the time $1/\omega$ of the electron transfer from one atomic core to another. Here v is the relative velocity of the nuclei and a is a characteristic size which can depend on the distance between the nuclei.

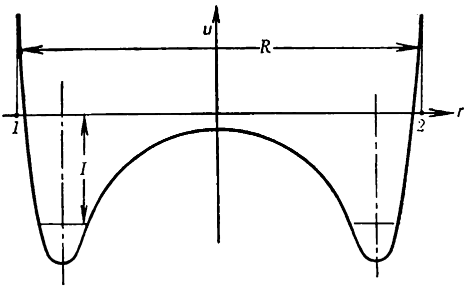


Fig. 9. Cross section of the equipotential surface for an electron in the field of two identical atomic cores. 1 and 2 are the positions of the nuclei of the cores, R is the distance between them, and I is the ionization potential, that is, the binding energy of the electron to one of the cores.

The frequency of transfer $\omega(R)$ is proportional to the probability of the potential barrier penetration by the electron; quantum mechanics gives $\omega(R) \sim \exp(-\gamma R)$ where $\gamma = (2mI/\hbar^2)^{1/2}$, I is the binding energy of the electron in the atomic core (the ionization potential of the atom), and m is the electron mass.

Hence, the main contribution to the cross section of the charge exchange is given by the electron transitions at certain distances R_0, which are determined by the collision velocity: $\exp(-\gamma R_0) \sim v$. This yields the cross section of the resonance charge exchange ($\sigma_{\text{res.}} \sim R_0^2$) as a function of the collision velocity:

$$\sigma_{\text{res.}} = \sigma_0 \ln^2 \frac{v_0}{v} \qquad (4.17)$$

where σ_0 and v_0 are certain parameters of the system. A similar velocity dependence is found for other resonance processes which do not involve elastic scattering and are accompanied by the exchange of one or two electrons. The cross section of the resonance charge exchange is seen to be only slightly dependent on the collision velocity. The same conclusion follows from Appendix 2 where the cross sections of the resonance charge exchange are given for specific atomic pairs. These data show that the cross section of the resonance charge exchange is larger than the gas-kinetic cross section.

5 Formation and Decomposition of Charged Particles in Weakly Ionized Gas

Since the electric characteristics of the weakly ionized gas are due to the charged particles, the processes of their formation and decomposition are of special importance.

The charged particles in the weakly ionized gas are produced in the collisions between the electrons and the atoms or molecules. Ionization by electron impact may occur in two ways according to the state of the gas. The ionization of the atom or molecule following one collision with the electron is termed the single ionization. When the collisions with the electrons make the atom or molecule pass a number of excited states and ionization occurs from an excited state, this ionization is called the multistage ionization. Ionization of this type occurs when the electron density is sufficiently high and the excited atom or molecule does not have time to decay between two successive collisions with the electrons.

The charged gas particles decompose in various ways. For instance, neutralization of the charged particles occurs at the walls of the vessel filled with the weakly ionized gas. Mutual neutralization of the charged particles (the collisions between electrons and positive ions or positive and negative ions) is termed the recombination of charged particles. This process is characterized by the recombination coefficient α, which appears in the balance equation for the density of charged particles:

$$\frac{dN_e}{dt} = -\alpha N_e N_i \qquad (5.1)$$

where N_e is the electron density and N_i is the ion density.

Ionization of an atom in a single collision with an electron. Let us consider the process of detachment of the atomic electron in collision between the atom and another electron:

$$e + A \rightarrow 2e + A^+ \qquad (5.2)$$

The incident electron interacts with the valence electron of the atom; between them there occurs an exchange of energy exceeding the ionization potential of the atom, that is, the binding energy for the valence electron in the atom. Let us analyze this process in terms of the simplest model developed first by J.J. Thomson. Assume that the valence electron is at rest, the motion of electrons is governed by the classical laws, and the electrons do not interact with the atomic core during the collision. Far from providing a quantitative description of the process, this model gives a correct qualitative explanation of it.

In the framework of this model, ionization occurs each time the valence electron receives energy exceeding the ionization potential I of the atom. Hence, we have to find the cross section for such collision in which the electrons exchange a given energy $\Delta\varepsilon$. Consider the process in which the energy transferred, $\Delta\varepsilon$, is much lower than the energy of the incident electron, $\mathscr{E} = mv^2/2$. Then the incident electron travels in a straight line, the valence electron may be assumed at rest during the collision, and the variation of the electron momentum is perpendicular to the path of the incident electron in the plane of motion.

Newton's equation for variation of the momentum of the incident electron is $\mathbf{F} = d\Delta\mathbf{p}/dt$ where $\mathbf{F} = e^2\mathbf{R}/R^3$ is the force with which the valence electron acts on the incident electron, and R is the distance between the electrons. Hence, the component of the momentum variation in the direction perpendicular to the path satisfies the equation $d\Delta p_\perp/dt = e^2\rho/R^3$ (ρ is the impact parameter of the collision; see Fig. 10) and this variation is

$$\Delta p_\perp = \int_{-\infty}^{+\infty} \frac{e^2\rho}{R^3}\,dt = \frac{2e^2}{\rho v}$$

Here we made use of the following relation for the free motion of the electron: $R^2 = \rho^2 + v^2t^2$. Now we can find the energy lost by the incident electron and received by the valence electron:

$$\Delta\varepsilon = \frac{\Delta p_{\perp}^{2}}{2m} = \frac{2e^{4}}{\rho^{2} m v^{2}} = \frac{e^{4}}{\rho^{2}\mathcal{E}}$$

where $\mathcal{E}$ is the energy of the incident electron. Hence, we find the following expression for the cross section of collisions accompanied by the exchange of the energy $\Delta\varepsilon$:

$$d\sigma = 2\pi\rho\, d\rho = \frac{\pi e^{4}\, d\Delta\varepsilon}{\mathcal{E}\,\Delta\varepsilon^{2}} \tag{5.3}$$

Owing to the method of its derivation, Eq. (5.3) is valid for $\Delta\varepsilon \ll \mathcal{E}$.* Now we shall extrapolate it to cover the range $\Delta\varepsilon \sim \mathcal{E}$. Ionization in the Thomson model occurs when the energy transferred, $\Delta\varepsilon$, exceeds the ionization potential of the atom, I; hence, we obtain the following expression for the cross section of atom ionization:

$$\sigma_{\text{ion}} = \int\limits_{\Delta\varepsilon \geqslant I} d\sigma = \frac{\pi e^{4}}{\mathcal{E}}\left(\frac{1}{I} - \frac{1}{\mathcal{E}}\right) \tag{5.4}$$

This expression for the ionization cross section is called the Thomson formula. The formula was derived for the atom with one valence electron. For an atom with several valence electrons the cross section (5.4) must be multiplied by their number.

Note that we can make directly some conclusions about the ionization cross section using classical concepts. Indeed, we can make use of the following classical parameters: the electron mass m, the interaction parameter e^{2}, the energy of the incident electron $\mathcal{E}$, and the ionization potential of the atom I. The most general form of the cross section expressed in terms of these parameters is

$$\sigma_{\text{ion}} = \frac{\pi e^{4}}{I^{2}} f\left(\frac{\mathcal{E}}{I}\right) \tag{5.5}$$

where $f(\mathcal{E}/I)$ is a universal function (that is, identical for all atoms). In particular, for the Thomson model, where additional assumptions are made, this function is

$$f(x) = \frac{1}{x} - \frac{1}{x^{2}}$$

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* Actually, Eq. (5.3) is valid over the whole range $\Delta\varepsilon \leqslant \mathcal{E}$

The Thomson formula gives a correct qualitative expression for the ionization cross sections. In the vicinity of the threshold, the ionization cross section is proportional to the energy; it exhibits a maximum when the energy of the incident electron is comparable to the ionization potential, and this maximum is of the same order of magnitude as the cross-sectional area of the atom. For high energies of the incident electron, the ionization cross section decreases with the reciprocal of the energy.

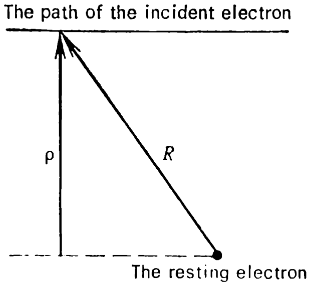

FIG. 10. Collision between an electron and a resting electron for a large impact parameter.

Recombination of pairs of positive and negative ions. This process can be represented by the following scheme:

$$A^+ + B^- \rightarrow A + B \tag{5.6}$$

The recombination consists in the valence electron going over from the field of the atom B to the field of the ion A^+. This tunnelling of the electron is effective when the distance R_0 between the ions is comparable to or somewhat larger than the size of the negative ion. Let us simplify the problem and assume that the electron is transferred when the distance of closest approach, r_0, is less or equal to R_0 and is not transferred for larger impact parameters. Using Eq. (4.5), we obtain the following expression for the cross section of the ion pair recombination:

$$\sigma_{\text{rec.}} = \pi\rho_0^2 = \pi R_0^2 [1 + e^2/(\varepsilon R_0)] \tag{5.7}$$

Here ρ_0 is the impact parameter for the collision in which the distance of closest approach is R_0. For low collision energies $\varepsilon \ll e^2/R_0$, Eq. (5.7) yields

$$\sigma_{\text{rec.}} = \pi R_0 \frac{e^2}{\varepsilon} \qquad (5.8)$$

Hence, we find the constant of the ion pair recombination $\alpha = v\sigma_{\text{rec.}} = \pi\sqrt{2}e^2R_0/(\varepsilon\mu)^{1/2}$. Averaging this quantity over the Maxwell distribution (2.15a) for the relative velocity of the ions yields

$$\alpha = \langle v\sigma_{\text{rec.}}\rangle = 2\sqrt{2\pi}\, R_0e^2/\sqrt{\mu T} \qquad (5.9)$$

where the angle brackets denote averaging over the ion velocities.

Triple processes. The processes represented by the scheme

$$A + B + C \rightarrow AB + C \qquad (5.10)$$

are referred to as the three-particle or triple processes. Among them are a number of processes described in Tables 1-3.

The triple process results in the binding of two colliding particles while the third particle carries away the energy released in the process.

The balance equation for the densities of the particles produced in the process (5.10) has the following form:

$$\frac{d[AB]}{dt} = \mathscr{K}[A][B][C] \qquad (5.11)$$

where $[X]$ is the density of the respective particles, and $\mathscr{K}$ is the constant of the triple process, which does not depend on the particle densities and has the dimensions cm^{6}s^{-1}. The triple process can involve a large number of pair collisions. Typically, the process (5.10) first gives rise to an excited state of the bound system AB, which later goes over to its ground state following multiple pair collisions.

Thomson's theory for the constant of the triple process. Let us follow Thomson's argument in evaluating the constant of the process (5.10) under the following conditions*: the energy required to break the A-B bond is considerably higher than the thermal energy of the particles, the motion of the particles is governed by classical laws, and formation of

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* Thomson developed his theory for the process $A^- + B^+ + C \rightarrow A + B + C$, but it can be readily extended to the general case.

the bound state of particles is due to their interaction. These conditions make it possible to estimate $\mathcal{K}$ for some triple processes and to determine the relationship between $\mathcal{K}$ and the parameters of the problem.

Formation of the bound state of the particles A and B occurs in the following way. When particles A and B approach each other, their kinetic energy increases since the potential energy of attraction between these particles converts into kinetic energy. Assume that when these particles are close by, a third particle C hits one of the two particles and takes from it some of its energy. If the energy transferred to the particle C is greater than the initial kinetic energy of particles A and B, the bound state of these particles is produced.

Our calculations will be based on the above considerations. Let us assume that the characteristic energies of the particles are of the order T and the mass of the particle C is comparable to the mass of one of the first two particles so that the characteristic energy exchange is of the order of the thermal energy of the particles, T. Since this energy exchange must exceed the initial kinetic energy of the particles A and B, the potential of the interaction between these particles during collision with the particle C must also be of the order of T. Thus, let us define the so-called critical radius b by the following relation:

$$U(b) \sim T \tag{5.12}$$

where U is the potential of the interaction between the particles A and B.

Now let us estimate $\mathcal{K}$, the constant of the triple process (5.10). The frequency of conversion of particle B into particle AB is equal, by order of magnitude, to the product of the probability for the particle B to be in the critical region $[B]\, b^3$ and the frequency $[C]\, v\sigma$ of collisions with particle C. Here v is the characteristic relative velocity of collision and σ is the cross section of the collisions between particle C and particles A or B, which result in the energy exchange of the order T. If the masses of the colliding particles are comparable, this cross section equals the cross section of the elastic collision of the particles.

This estimate yields the following expression for the rate of production of particles AB:

$$\frac{d[AB]}{dt} \sim [A][B]\, b^3 [C]\, v\sigma$$

A comparison of this expression with the definition of the constant of the triple process (5.11) yields the following estimate for $\mathscr{K}$:

$$\mathscr{K} \sim b^3 v\sigma \qquad (5.13)$$

The process (5.10) is a triple process if the density $[C]$ of particles C is small and hence there is a low probability that the colliding particles A and B will interact with particle C in the critical region. This condition is satisfied if the critical radius b is small compared to the mean free path $\lambda \sim ([C]\,\sigma)^{-1}$ of particle A or B:

$$[C]\,\sigma b \ll 1 \qquad (5.14)$$

If this condition is not satisfied, the process (5.10) is not a triple process and Eq. (5.13) is inapplicable.

Triple recombination of electrons and ions. The process of triple recombination of electrons and positive ions is important for high-density plasma; it can be represented by the following scheme:

$$2e + A^+ \rightarrow A + e \qquad (5.15)$$

In the framework of the Thomson theory, the triple collision produces initially an excited atom whose ionization potential is of the order of the thermal energy T, and later this atom goes over to the ground state following multiple collisions with electrons. The Thomson theory can be applied to this process since the classical laws are valid for the highly excited state of the atom ($T \ll I$, where I is the ionization potential of the atom).

Let us evaluate the constant of the process (5.15) proceeding from Eq. (5.13). According to Eq. (4.14), the cross section of the elastic collision of electrons is $\sigma \sim e^4/T^2$, and according to Eq. (5.12), the critical radius for the Coulomb interaction between an electron and an ion is $b \sim e^2/T$; hence, the constant of the process (5.15) is

$$\mathscr{K} = \frac{\alpha}{N_e} \sim \frac{e^{10}}{m^{1/2}T^{9/2}} \qquad (5.16)$$

where α is the recombination coefficient defined by Eq. (5.1). Equation (5.16) is valid if the condition (5.14) is satisfied; in this case this condition coincides with the condition of plasma ideality (3.1) and has the following form:

$$N_e e^6/T^3 \ll 1$$

Equation (5.16) presents a correct relationship between the recombination coefficient and the parameters of the problem. This relationship can be derived in a simpler way using dimensional analysis. Indeed, for the suggested mechanism of the process (5.15), its constant depends only on the parameter of the interaction e^2, the electron mass m, and the thermal energy T (the temperature of the electrons). There is only one combination of these three parameters which has the dimensions cm^6s^{-1} of the constant of the triple process; it can be readily seen that this combination coincides with Eq. (5.16).

A similar procedure can be used to determine the constant of the process of formation of the molecular ions from the atomic ions by the following scheme:

$$A^+ + 2A \rightarrow A_2^+ + A \qquad (5.17)$$

This process is due to the polarization interaction between the ion and the atom; the potential of this interaction is $U(R) = -\beta e^2/2R^4$ where R is the distance between the ion and the atom, and β is the polarizability of the atom. Only the combination of the interaction parameter βe^2, the mass of the nucleus M, and the thermal energy of the particles T has the dimension of $\mathcal{K}$; this combination is

$$\mathcal{K} \sim \frac{(\beta e^2)^{5/4}}{M^{1/2}T^{3/4}} \qquad (5.18)$$

The same expression for the constant for the process (5.17) can be derived directly from the Thomson formula (5.13).

Triple recombination of the positive and negative ions. The process of triple recombination of ions in triple collisions may be described by the following scheme:

$$A^+ + B^- + C \rightarrow A + B + C \qquad (5.19)$$

The process gives rise initially to a bound state of the positive and negative ion A^+-B^-; after a while a valence electron is transferred from the field of the atom B to the field of the ion A^+ and the bound state decays into two atoms A and B.

The second stage of this process occurs spontaneously following the capture of the negative ion by the positive ion so that the triple recombination is determined by the formation of the bound state of these ions. Hence, let us estimate the constant for the process (5.19) assuming the Coulomb interaction between the ions and the polarization interaction between the ion and the atom. If the atom mass M is comparable to the mass of one of the ions, Eq. (4.12) for the cross section of the atom-ion collision and Thomson's formula (5.13) yield

$$\mathcal{K} = \frac{\alpha}{N_i} \sim \frac{e^6}{T^3}\left(\frac{\beta e^2}{M}\right)^{1/2} \qquad (5.20)$$

Equation (5.20) is valid if the condition (5.14) is satisfied, which in our case means that

$$[C]\,\frac{e^2\,(\beta e^2)^{1/2}}{T^{3/2}} \ll 1 \qquad (5.21)$$

Inserting into Eq. (5.21) the polarizability of the atom, which is about several atomic units, we find that at room temperature $[C]$ is much less than 10^{20} cm^{-3}, that is, the process of triple recombination of positive and negative ions is disrupted in this case at the gas pressure of about one atmosphere.

Processes involving formation of a long-lived complex. If collision of particles produces unstable bound states with large lifetimes, the constant of the triple process can prove to be larger than the value found from Thomson's theory. Such continuous spectrum bound states are termed auto-ionization states. An auto-ionization state can appear when the excess kinetic energy of the particle is transferred to the internal degrees of freedom. For instance, in the interaction between the ion and the electron the bound state of the particles can be produced by exciting the inner or valence electrons of the ion and forming a bound state of the incident electron and the ion. If the energy of the incident electron is less than the excitation energy for the inner electrons of the ion, such a system can decompose into an ion and a free electron owing to transfer of the excitation energy to an inner electron of the ion.

In a similar way, we can produce the auto-ionization state of a molecule with an ion or atom by transferring the excess energy to an excited vibration-rotational state of the molecule. There are other types of the auto-ionization states, and all such states are the bound states of particles whose discrete energy levels lie in the continuous spectrum. The lifetimes of the auto-ionization states are great compared to the characteristic times for such systems, that is, the times in which the particles pass the interaction region. This is due to the fact that decay of the auto-ionization state involves transition between various degrees of freedom. It is the large lifetimes of the auto-ionization states that determines their significance for production of the bound states of colliding particles.

Formation of the bound states of particles via the auto-ionization states can be described by the following scheme:

$$A+B\xrightarrow{k_{\text{capture}}}AB^{**}, \qquad (5.22a)$$

$$AB^{**}\xrightarrow{\tau}A+B, \qquad (5.22b)$$

$$AB^{**}+C\xrightarrow{k_{\text{quench.}}}AB+C, \qquad (5.22c)$$

$$AB^{**}+C\xrightarrow{k_{\text{decomp.}}}A+B+C \qquad (5.22d)$$

Here AB^{**} is the auto-ionization state of the particles; the quantities written over the arrows are the constants for the respective processes and the lifetime of the auto-ionization state AB^{**}. Hence, the balance equation for the density of the auto-ionization states has the following form:

$$\frac{d[AB^{**}]}{dt}=0=[A]\,[B]\,k_{\text{capture}}-\frac{[AB^{**}]}{\tau}$$
$$-[AB^{**}]\,[C]\,(k_{\text{quench.}}+k_{\text{decomp.}})$$

The equation yields

$$[AB^{**}]=\frac{[A]\,[B]\,k_{\text{capture}}}{\tau^{-1}+[C]\,(k_{\text{decomp.}}+k_{\text{quench.}})}$$

The equation for the process (5.22c) is

$$\frac{d[AB]}{dt} = k_{\text{quench.}}[AB^{**}][C] = [A][B][C] \times \frac{k_{\text{capture}}k_{\text{quench.}}\tau}{1+[C](k_{\text{quench.}}+k_{\text{decomp.}})\tau}$$

A comparison of this balance equation and the definition ot the constant of the triple process (5.13) shows that this process is a triple process if the condition

$$[C](k_{\text{quench.}}+k_{\text{decomp.}})\tau \ll 1 \qquad (5.23\text{a})$$

is satisfied, and the constant of the triple process in this case is given by

$$\mathscr{K} = k_{\text{capture}}k_{\text{quench.}}\tau \qquad (5.23\text{b})$$

For another limiting case

$$[C](k_{\text{quench.}}+k_{\text{decomp.}})\tau \gg 1 \qquad (5.24\text{a})$$

the density of the particles C is so high that the rate of production of the bound state AB does not depend on $[C]$ and we have

$$\mathscr{K}[C] = k_{\text{capture}}\frac{k_{\text{quench.}}}{k_{\text{quench.}}+k_{\text{decomp.}}} \qquad (5.24\text{b})$$

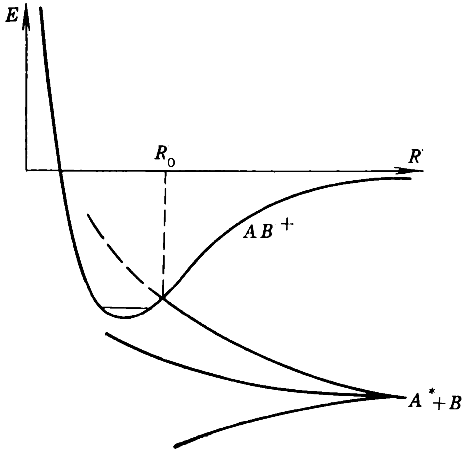
The particles C affect the factor $k_{\text{quench.}}/(k_{\text{quench.}}+k_{\text{decomp.}})$ in Eq. (5.24b); this factor is the probability that the collision between particle C and particle AB in the auto-ionization state will force AB to transfer to the stable bound state.

The lifetime of the auto-ionization state increases sharply with an increase in the number of the internal degrees of freedom since the excess energy can be distributed over a larger number of the degrees of freedom. The auto-ionization state produced in collision of systems with large numbers of degrees of freedom is called a long-lived complex. It is the formation of a long-lived complex that makes it possible to satisfy the condition (5.24a). For instance, the lifetime of the long-lived complex made up of large molecules and electrons is of the order of 10^{-6}-10^{-4} s, that is, the condition (5.24a) is satisfied for $[C] \gg 10^{15}$-10^{17} cm^{-3}. Typically, the lifetime of the long-lived complex is considerably smaller and the process has a triple character. Note that we can obtain Thomson's formula (5.13) by replacing τ in Eq. (5.23b) with the transit time of the interacting particles.

Dissociative recombination of an electron and a molecular ion. This process can be described by the following scheme:

$$e + AB^+ \rightarrow A^* + B \qquad (5.25)$$

FIG. 11. The terms of a molecular ion and an excited molecule involved in dissociative recombination.



The high importance of the process is due to the fact that the coefficient of dissociative recombination does not depend on the densities of the particles and the rates of recombination are relatively high.

Figure 11 illustrates the mechanism of dissociative recombination. It shows the term of the ground state of the molecular ion and also one of the terms of the molecule consisting of atom B and atom A^* in an excited state. These terms intersect for a certain distance R_0 between the nuclei so that for smaller distances the state A^*B is an auto-ionization state. It is this auto-ionization state that makes possible dissociative recombination.

In the course of dissociative recombination, the electron colliding with the molecular ion is captured to the auto-ionization level. Since the interaction between the atoms in the auto-ionization state is repulsive, they move apart. If the auto-ionization state does not have time to decay while the atoms move apart to the distance R_0, the result is a stable state of the particles, that is, there occurs recombination of the electron and the molecular ion giving rise to the excited atom and the atom in the ground state.

Dissociative recombination is a fairly complicated process since the number of the auto-ionization states of the system A^*-B is large (sometimes, infinite) and the lifetime of these states (with respect to decomposition into an electron and a molecular ion) depends on the distance between the nuclei. Moreover, the recombining molecular ion can be in excited vibrational states and this fact also influences the magnitude of the recombination coefficient. Owing to the above difficulties the general expressions for the recombination coefficient can hardly be derived, while the expressions for special cases which are highly involved carry insufficient information and we shall not discuss them here. Note that when the conditions for dissociative recombination are satisfied, the recombination coefficient has the same or higher order of magnitude as the respective quantity for atoms.

Let us consider a simple model of dissociative recombination of an electron and a complex molecular ion. The latter has a large number of internal degrees of freedom which can absorb the excess energy. Therefore, any capture of an electron on the auto-ionization level can be assumed to result automatically in recombination, and the electron is captured if it gets into the region of strong interaction with the molecular ion, that is, into the region where this ion is located.

This model can be used for estimating the coefficient of dissociative recombination of an electron and a complex molecular ion. Let us assume that the behaviour of the electron is governed by classical laws and that if it gets into the region of the size R_0, the recombination will occur. This model was used above in the analysis of the ion pair recombination and it yielded Eq. (5.9), which in this case gives

$$\alpha \approx 5R_0e^2/\sqrt{mT} \qquad (5.26)$$

where R_0 is of the order of the size of the molecular ion, and m is the electron mass.

Equation (5.26) can be used for estimating the coefficient of dissociative recombination. At room temperature the coefficient of dissociative recombination of an electron and a complex molecular ion is of the order of 10^{-6} cm^{3}s^{-1}, which agrees with experimental results.

Ionization processes in collisions between an atom in an excited state and an atom in the ground state. Associative ionization is the process opposite to dissociative recombination and can be described by the following scheme:

$$A^* + B \rightarrow AB^+ + e \tag{5.27}$$

The colliding atoms occur on the term which intersects the molecular ion term (see Fig. 11), and the respective state is the auto-ionization state for the distances between the nuclei less than R_0. It is this state that decomposes into a molecular ion and an electron. The cross section of associative ionization depends on the lifetime of the auto-ionization state. Hence, at low energies only a limited number of the excited states have significant cross sections of associative ionization. This process is important for production of charged particles in plasma.

Another practically important ionization process in collisions of atoms is the Penning effect, which can be described by the following scheme:

$$A^* + B \rightarrow A + B^+ + e \tag{5.28}$$

where A^* is a metastable atom whose excitation energy is higher than the ionization potential of atom B. The metastable atomic state is an excited state which does not decay or which decays slowly due to radiation transition. Such metastable states are found, for instance, in atoms of inert gases or atoms of alkali-earth metals but not in atoms of alkali metals. Since the metastable atoms have large lifetimes, their concentration in the weakly ionized gas can be high and exceed the concentration of the charged particles. Hence, the rate of production of charged particles is changed, owing to the Penning effect, by addition of an easily ionized impurity to the weakly ionized gas even in such small quantities as a few per cent of the density of the gas.

The Penning effect is essentially the decomposition of the auto-ionization state of system A^*B. The initial state of the system is the auto-ionization state for any distance between the nuclei. However, for large distances the lifetime of this state is large because of the weak interaction in the system. The lifetime of the auto-ionization state decreases with decreasing distance between the nuclei, and the Penning effect occurs if this state is able to decay during the collision.

Stepwise ionization of atoms. In the gas-discharge plasma, the mean electron energy is usually considerably lower than the ionization potential of the atoms. Under these circumstances single ionization of atoms can occur only in collisions with the high-energy electrons from the tail of the distribution function. Ionization can occur also when the atom passes a number of excited states in collisions with electrons and is ionized from an excited state; this process is termed stepwise ionization. Let us find the constant of stepwise ionization assuming that the electrons have the Maxwell velocity distribution and the electron temperature is considerably lower than the ionization potential of the atom:

$$T_e \ll I \tag{5.29}$$

Assume that the electron density in the system is sufficiently high and all the transitions between the excited states of the atoms are due to collisions with the electrons. Under these assumptions stepwise ionization proves to be the detailed reverse of the process of triple recombination of an electron and ion (5.15). In these processes the atoms undergo the same transformations but in the opposite directions.

Assume that the electrons in the system are in thermodynamic equilibrium with the atoms. Under these conditions the charged particles are produced as a result of stepwise ionization while their decay is due to triple recombination; hence, the balance equation for the electron density is

$$\frac{dN_e}{dt} = 0 = N_e N_a k_{\text{step.}} - \alpha N_e N_i$$

Here N_a is the atom density and $k_{\text{step.}}$ is the constant of stepwise ionization of the atoms. Since the electrons and atoms are in thermodynamic equilibrium and the Saha distribution (2.17) gives the relationship between their densities, we obtain the following relationship between the constants of the above processes:

$$k_{\text{step.}} = \frac{\alpha}{N_e} \frac{g_e g_i}{g_a} \left(\frac{mT}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{I}{T_e}\right) \tag{5.30}$$

where g_e, g_i and g_a are the statistical weights of the electron, ion and atom, respectively, and T_e is the electron temperature.

It should be noted that relationship (5.30) is valid for any ratio between the atom and electron densities since the constants $k_{\text{step.}}$ and α/N_e do not depend on the densities of the particles. The assumption of the thermodynamic equilibrium in the system was made only to help us find the relationship between the constants.

Inserting Eq. (5.16) for the triple recombination coefficient into Eq. (5.30), we obtain the following expression for the constant of stepwise ionization:

$$k_{\text{step.}} = A \frac{g_i}{g_a} \frac{me^{10}}{\hbar^3 T_e^3} \exp\left(-\frac{I}{T_e}\right) \qquad (5.31)$$

where A is a numerical factor identical for all the atoms. Let us compare the constant of stepwise ionization with the constant of single ionization, which is given in this case by

$$k_{\text{ion}} = \int_I^\infty \frac{2\mathscr{E}^{1/2}}{\pi^{1/2} T_e^{3/2}} \exp\left(-\frac{\mathscr{E}}{T_e}\right) \left(\frac{2\mathscr{E}}{m}\right)^{1/2} \sigma_{\text{ion}}(\mathscr{E})\, d\mathscr{E}$$

where $\mathscr{E}$ is the energy of the incident electron. If the condition (5.29) is satisfied, this integral converges in the vicinity of the ionization potential where the cross section is given by $\sigma_{\text{ion}} = \sigma_0 (\mathscr{E}/I - 1)$; here σ_0 is of the order of the atomic cross section. Taking into account the condition (5.29) and using this cross section, we find

$$k_{\text{ion}} = \left(\frac{8T_e}{\pi m}\right)^{1/2} \sigma_0 \exp\left(-\frac{I}{T_e}\right) \qquad (5.32)$$

A comparison of Eqs. (5.31) and (5.32) yields ($\sigma_0 \sim a_0^2$ where a_0 is the Bohr radius)

$$\frac{k_{\text{ion}}}{k_{\text{step.}}} \sim \left(\frac{\hbar^2 T_e}{me^4}\right)^{7/2}$$

This ratio is much less than unity owing to the condition (5.29) and to the fact that the ionization potential of the atom, I, is of the order of $me^4/\hbar^2$. Hence, if the conditions are suitable for stepwise ionization, this process at low electron temperatures proves to be more efficient than the single ionization of an atom.

Thermodynamic equilibrium conditions for excited atoms. When deriving Eq. (5.31), we made an implicit assumption that the excited atoms involved in ionization are in thermodynamic equilibrium with the atoms in the ground state. Let us find out the conditions under which this assumption is valid.

Let us consider two states of the atom, the ground state which we denote by the subscript "gr.", and the excited state denoted by "exc.". A transition between these two states can be due to collisions with electrons; the excitation constant averaged over the electron velocities will be denoted by $k_{\text{gr.exc.}}$, the constant for the opposite transition by $k_{\text{exc.gr.}}$. Apart from that, there are other mechanisms of decay of excited states (radiation, decay at the walls, etc.) for which the lifetime is τ. Let us determine the condition of thermodynamic equilibrium under these circumstances.

The balance equation for the density of excited atoms in our case is

$$\frac{dN_{\text{exc.}}}{dt}=0=N_{\text{gr.}}N_e k_{\text{gr.exc.}}-N_{\text{exc.}}N_e k_{\text{exc.gr.}}-\frac{N_{\text{exc.}}}{\tau} \qquad (5.33)$$

First, we shall consider the plasma in which the transitions between the states occur in collisions with the electrons, that is, $\tau \rightarrow \infty$. Then the density of the excited atoms is given by the Boltzmann distribution (2.9):

$$N_{\text{exc.}}=N_{\text{gr.}}\frac{g_{\text{exc.}}}{g_{\text{gr.}}}\exp\left(-\frac{\Delta\mathscr{E}}{T_e}\right)$$

Hence, we obtain the following relationship between the constants:

$$k_{\text{gr.exc.}}=k_{\text{exc.gr.}}\frac{g_{\text{exc.}}}{g_{\text{gr.}}}\exp\left(-\frac{\Delta\mathscr{E}}{T_e}\right) \qquad (5.34)$$

where $\Delta\mathscr{E}$ is the energy difference between the states, and T_e is the electron temperature. Since $k_{\text{gr.exc.}}$ and $k_{\text{exc.gr.}}$ do not depend on the density of the atoms, relationship

(5.34) is valid when the thermodynamic equilibrium between the atom states is disrupted. This relationship follows directly from the principle of detailed balancing.

The balance equation (5.33) and Eq. (5.34) can be used to determine the density of the excited atoms:

$$N_{\text{exc.}} = N_{\text{gr.}} \frac{k_{\text{gr.exc.}}}{k_{\text{exc.gr.}}} \left(1 + \frac{1}{N_e k_{\text{exc.gr.}} \tau}\right)^{-1}$$
$$= N_{\text{gr.}} \frac{g_{\text{exc.}}}{g_{\text{gr.}}} \exp\left(-\frac{\Delta \mathscr{E}}{T_e}\right) \left(1 + \frac{1}{N_e k_{\text{exc.gr.}} \tau}\right)^{-1}$$

It can be readily seen that our system is in thermodynamic equilibrium when

$$N_e k_{\text{exc.gr.}} \tau \gg 1 \qquad (5.35)$$

that is, when the time of decay of a given excited state via the collision with an electron is considerably less than the time of its decay via other decay channels. This condition is satisfied at high electron densities. The lifetime of excited atoms with respect to radiative transition is larger for higher excited levels. Therefore, condition (5.35) is not satisfied, typically, for the lower excited states of atoms while the higher excited states are in thermodynamic equilibrium.*

Now let us discuss once again Eq. (5.31) and determine how it can be applied taking into account the radiative transitions in the atoms and also the decay of the excited atoms at the walls of the vessel filled with plasma. Equation (5.31) is valid if radiative transition does not affect the process, that is, if condition (5.35) is satisfied for the excited atoms (here τ is the lifetime of the excited states with respect to radiative transitions). Actually, this condition should be satisfied only for the lower excited states. Moreover, the excited atom must have time to undergo all the transformations leading to ionization long before it

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* This means that the Boltzmann distribution (2.9) gives the relationship between the densities of the atomic excited states i and k, the transitions between which occur owing to collisions with the electrons. These excited atoms may be not in thermodynamic equilibrium either with the free electrons or the atoms in the ground state if there is no general thermodynamic equilibrium.

can drift to the wall of the vessel. Hence,

$$N_e k_{\text{step.}} \tau_{\text{drift}} \gg 1$$

where τ_{drift} is the characteristic time for the atomic drift to the wall.

6 Physical Kinetics of Gas and Plasma

The kinetic equation. We have to give the mathematical description of the system of particles comprising the gas. Though collisions in the gas are rare and each particle spends only a small fraction of its time, $N\sigma^{3/2}$, strongly interacting with other particles (see Sec. 4), the significance of these collisions is that they result in energy exchange between the particles. Hence, a description of the system behaviour must take into account collisions of the particles though we can confine ourselves only to the pair collisions, since the probability of the triple collisions is by a factor of $N\sigma^{3/2}$ less than the probability of the pair collisions while the probability of collisions between a larger number of particles is even lower.

Now we have to choose a parameter for describing the system. The state of an individual particle is described by the velocity $\mathbf{v}$ and the inner quantum numbers J. We have to find at each moment of time how many particles in each state are at each point in space. Hence, we describe the system by a distribution function $f(\mathbf{v}, J, \mathbf{r}, t)$ so that $f(\mathbf{v}, J, \mathbf{r}, t)\, d\mathbf{v}$ is the number of particles in unit volume at point $\mathbf{r}$ at moment t which have the inner quantum numbers J and the velocities in the range from $\mathbf{v}$ to $\mathbf{v} + d\mathbf{v}$. Hence, the density of particles at point $\mathbf{r}$ at moment t is, by definition,

$$N(\mathbf{r}, t) = \sum_J \int f(\mathbf{v}, J, \mathbf{r}, t)\, d\mathbf{v} \qquad (6.1)$$

The evolution of the system can be described by the equation which is satisfied by the distribution function $f(\mathbf{v}, J, \mathbf{r}, t)$ and which is called the kinetic equation. We assume in this equation that the variation of the number of particles in a given state is due to the pair collisions between the particles. The equation can be written as

$$\frac{df}{dt}=I_{\text{col.}} \tag{6.2}$$

where $I_{\text{col.}}$ is the so-called collision integral, which takes into account the variation of the number of particles in a given state owing to collisions with other particles.

Let us write down the left-hand side of the kinetic equation which describes the motion of particles in external fields in the absence of collisions

$$\frac{df}{dt}=\frac{f(\mathbf{v}+d\mathbf{v},\ \mathbf{r}+d\mathbf{r},\ t+dt)-f(\mathbf{v},\ \mathbf{r},\ t)}{dt}$$

In the absence of collisions we have $d\mathbf{v}/dt=\mathbf{F}/M$ where $\mathbf{F}$ is the force of the external field acting on the particle, M is the mass of the particle, and $d\mathbf{r}/dt=\mathbf{v}$. Hence,

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial \mathbf{r}}+\frac{\mathbf{F}}{M}\frac{\partial f}{\partial \mathbf{v}}$$

so that the kinetic equation is transformed into

$$\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial \mathbf{r}}+\frac{\mathbf{F}}{M}\frac{\partial f}{\partial \mathbf{v}}=I_{\text{col.}}(f) \tag{6.3}$$

Equation (6.3) describes the development of the system in time and is called the Boltzmann kinetic equation.

Let us consider the collision integral. It characterizes the variation of the distribution function owing to collisions between the particles. The collision integral's dimensions are f/τ where f is the distribution function, and τ is time. The characteristic time τ for variation of the distribution function owing to collisions is of the order of the time of collisions between the particles, $(Nv\sigma)^{-1}$, where N is the density of the particles, v is the characteristic velocity of collision, and σ is the cross section of the large-angle collisions. The collision integral is zero if the velocity distribution function is the Maxwell function f_0.

Proceeding from the above properties of the collision integral, we can derive the following simple approximation:

$$I_{\text{col.}}=-\frac{f-f_0}{\tau} \tag{6.4}$$

where f is the distribution function, and the characteristic time τ is of the order of the time between collisions and depends on the velocity. Approximation (6.4) is called the "tau" approximation; it yields approximate solutions of the kinetic equation which give a correct form of the distribution function.

Let us illustrate our discussion by applying the tau approximation to study the evolution of the system returning to the equilibrium state following a disturbance. There are no external fields and the system is uniform, so we obtain for the kinetic equation (6.3) in the tau approximation the following form:

$$\frac{\partial f}{\partial t} = -\frac{f-f_0}{\tau}$$

The solution of this equation is

$$f = f_0 + [f(\mathbf{v}, 0) - f_0] \exp(-t/\tau) \qquad (6.5)$$

where t is time, $f(\mathbf{v}, 0)$ is the velocity distribution function at zero time, and f_0 is the Maxwell distribution function. Relaxation of the system to its equilibrium occurs during a period of time which is of the order of the time between two consecutive collisions of the particles.

Macroscopic equations for a gas. The distribution function yields detailed information about a system of particles and can be used to express all the macroscopic parameters of the system. Hence, the kinetic equation may be used to derive all the equations for these macroscopic parameters. Below we shall derive the simplest macroscopic equations from the kinetic equation.

Let us integrate the kinetic equation (6.3) over the velocities of the particles. The right-hand side of the equation is the total variation of the density of particles per unit time due to collisions. Assuming that in the volume considered there is no production or decomposition of particles, we find that the right-hand side of the resulting equation equals zero:

$$\int \frac{\partial f}{\partial t}\,d\mathbf{v} + \int \mathbf{v}\,\frac{\partial f}{\partial \mathbf{r}}\,d\mathbf{v} + \frac{\mathbf{F}}{M}\int \frac{\partial f}{\partial \mathbf{v}}\,d\mathbf{v} = 0$$

Let us change the order of differentiation and integration in the first and second terms and make use of the fact that $\int f\,d\mathbf{v} = N$ and $\int \mathbf{v} f\,d\mathbf{v} = N\mathbf{w}$ where N is the density of the particles, and $\mathbf{w}$ is their mean velocity referred to as the drift velocity. The third term is zero since the distribution function for the infinite velocity is zero. Hence, we obtain

$$\frac{\partial N}{\partial t} + \operatorname{div}(N\mathbf{w}) = 0 \qquad (6.6)$$

This equation is termed the continuity equation.

To derive another equation, we shall multiply the kinetic equation by Mv_i and integrate over the velocities of the particles (M is the particle mass and v_i is the respective component of the particle's velocity, $i = x, y, z$). The right-hand side of the resulting equation is the variation of the total momentum of the particles due to collisions per unit time. Let us consider a system of identical particles. In this system, collisions do not change the total momentum of the particles so that the right-hand side of the equation is zero and the macroscopic equation has the following form:

$$\int Mv_i \frac{\partial f}{\partial t} d\mathbf{v} + \int Mv_i v_k \frac{\partial f}{\partial x_k} d\mathbf{v} + F_k \int v_i \frac{\partial f}{\partial v_k} d\mathbf{v} = 0$$

Here the subscripts i and k denote the components of the appropriate vector ($i, k = x, y, z$) with summation over k, and x_k is a coordinate. Let us change the order of integration and summation in the first two terms and integrate the third term by parts:

$$\int d\mathbf{v}\, v_i \frac{\partial f}{\partial v_k} = v_i f \Big|_{-\infty}^{+\infty} - \int dv f \delta_{ik} = -N\delta_{ik}$$

where $\delta_{ik} = 1$ when $i = k$ and $\delta_{ik} = 0$ when $i \neq k$. Finally, we obtain

$$\frac{\partial}{\partial t}(MNw_i) + \frac{\partial}{\partial x_k}(N\langle Mv_i v_k\rangle) - NF_i = 0$$

where the angle brackets denote averaging over the distribution of particles.

Let us define the pressure tensor as

$$\mathcal{P}_{ik} = \langle M(v_i - w_i)(v_k - w_k)\rangle \qquad (6.7)$$

Inserting this tensor into the above equation, we find that

$$\frac{\partial}{\partial t}(MNw_i) + \frac{\partial \mathcal{P}_{ik}}{\partial x_k} + \frac{\partial}{\partial x_k}(MNw_i w_k) - F_i = 0$$

Subtract from this equation the continuity equation $Mw_i\left[\frac{\partial N}{\partial t}+\frac{\partial}{\partial x_k}(Nw_k)\right]=0$. Finally, we obtain

$$M\frac{\partial w_i}{\partial t}+\frac{1}{N}\frac{\partial \mathcal{P}_{ik}}{\partial x_k}+Mw_k\frac{\partial w_i}{\partial x_k}-F_i=0 \tag{6.8}$$

The form of this equation for the mean momentum depends on the representation of the pressure tensor, which is determined by the properties of the system. We shall discuss below a special form of representation for this tensor. The macroscopic equation for the mean energy has an even more complicated form.

The macroscopic equations (6.6) and (6.8) have been derived for a one-component gas. Let us transform these equations to apply to a multicomponent system. The right-hand side of the continuity equation is the variation of the number of particles of a given species in unit volume per unit time due to production or loss of the particles of this species. In the absence of production or loss of this species, the continuity equation for this species has the form (6.6) irrespective of the presence or absence of other species in the system.

The right-hand side of Eq. (6.8) for the multicomponent system should contain the variation of the momentum of a given species per unit time owing to collisions with other species. If the mean velocities of two species are different, there occurs the momentum transfer between them. Hence, the momentum transfer between two species is proportional to the difference between their mean velocities. Therefore, we can write Eq. (6.8) in the following form:

$$\frac{\partial w_i^{(q)}}{\partial t}+\frac{1}{M_qN_q}\frac{\partial \mathcal{P}_{ik}^{(q)}}{\partial x_k}+w_k^{(q)}\frac{\partial w_i^{(q)}}{\partial x_k}-\frac{F_i^{(q)}}{M_q}=\sum_s\frac{w_i^{(s)}-w_i^{(q)}}{\tau_{qs}} \tag{6.9}$$

The subscripts s and q denote here the particle species, and τ_{qs} is the characteristic time of the momentum transfer from the species q to the species s. Since this transfer does not change the total momentum of the system, the characteristic time of the momentum transfer satisfies the following equation:

$$\frac{M_q N_q}{\tau_{qs}} = \frac{M_s N_s}{\tau_{sq}} \qquad (6.10)$$

The equation of state for a gas. The relationship between the macroscopic parameters of the gas (the pressure p, the temperature T, and the particle density N) is given by the equation of state. Let us derive this equation for a homogeneous gas. Let us express the gas pressure in terms of the distribution function. Let us introduce a new coordinate system in which the gas (or a given gas volume) is resting. In this coordinate system, the pressure is the force acting on a unit area of an imaginary surface in the system.

The flux of particles with the velocities in the range from v_x to $v_x + dv_x$ through the given unit area is $dj = v_x f\, dv_x$ with the x axis being perpendicular to the area. Reflection of each particle from the area is elastic, that is, the normal component of its velocity is reversed: $v_x \rightarrow -v_x$. Hence, the particle of the mass M transfers to the area a momentum of $2Mv_x$. The force acting on the area is the variation of the momentum per unit time. Hence, the gas pressure is given by

$$p = \int_{v_x > 0} 2Mv_x \times v_x f\, dv_x = M \int v_x^2 f\, dv_x = MN \langle v_x^2 \rangle$$

The integration in the first term is over the velocities v_x that exceed zero; it should be taken into account that the gas pressures on both sides of the area are the same.

In the above formula, v_x is the particle velocity component in the frame of reference where the gas is resting as a whole. The gas pressure in the general case is given by

$$p = MN \langle (v_x - w_x)^2 \rangle \qquad (6.11)$$

where w_x is the component of the mean velocity. In the coordinate system in which the gas is resting the distribution function for an isotropic gas $f(v_x)$ does not depend on direction. Under these circumstances the gas pressure is the same in all directions:

$$p = MN \langle (v_x - w_x)^2 \rangle = MN \langle (v_y - w_y)^2 \rangle = MN \langle (v_z - w_z)^2 \rangle \qquad (6.12)$$

Here we have a simple relationship between the pressure tensor (6.7) and the gas pressure:

$$\mathcal{P}_{ik} = p\delta_{ik} \qquad (6.13)$$

The definition of the gas temperature (2.16) relates the temperature to the mean kinetic energy of particles in the frame of reference in which the mean velocity is zero; hence, we obtain

$$3T/2 = M\langle(\mathbf{v} - \mathbf{w})^2\rangle/2$$

Using Eq. (6.12), we find the following relationship between the pressure and the temperature:

$$p = NT \qquad (6.14)$$

Equation (6.14) is known as the equation of state for a gas.

Let us write the macroscopic equation (6.8) for the mean momentum in the case of the isotropic gas when the pressure tensor has the form (6.13).* Inserting Eq. (6.13) into Eq. (6.8), we obtain

$$\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{w}\nabla)\,\mathbf{w} + \frac{\operatorname{grad} p}{\rho} - \frac{\mathbf{F}}{M} = 0 \qquad (6.15)$$

Here $\mathbf{w}$ is the mean velocity of the gas, $\rho = MN$ is the mass density, and $\mathbf{F}$ is the force acting on one particle. In the absence of external fields ($\mathbf{F} = 0$), this equation is called the Euler equation. Equation (6.9) for a multicomponent system is transformed in this case into

$$\frac{\partial \mathbf{w}_q}{\partial t} + (\mathbf{w}_q\nabla)\,\mathbf{w}_q + \frac{\operatorname{grad} p_q}{\rho_q} - \frac{\mathbf{F}_q}{M_q} = \sum_s \frac{\mathbf{w}_s - \mathbf{w}_q}{\tau_{qs}} \qquad (6.16)$$

where the subscripts q and s denote the gas particle species.

7 Transport Phenomena in Weakly Ionized Gas

Transport phenomena in gas and plasma. Transport of particles in a gas. The transport phenomena in gas and plasma primarily are due to the gradients of the macroscopic param-

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* Since Eq. (6.13) is valid also for the isotropic liquid, the equation derived below describes not only a gas but a liquid as well.

eters. These gradients give rise to fluxes that finally equalize the macroscopic parameters over the gas or plasma volume.

Let us consider the simplest transport phenomenon due to the transport of particles. If the density of a given particle species is not spatially uniform and this gas state is hydrodynamically stable, there is a directional flow of particles tending to equalize in space the density of the given particle species. If the density of the given species varies slightly over the mean free path, that is, the density gradient is small, the diffusion flux density $\mathbf{j}$ is proportional to that gradient:

$$\mathbf{j} = -\mathscr{D} \operatorname{grad} N \qquad (7.1)$$

The factor $\mathscr{D}$ in Eq. (7.1) is called the diffusion coefficient.

Let us estimate the magnitude of the diffusion coefficient. The net diffusion flux density equals the difference of the flux densities in opposite directions; each is, by order of magnitude, equal to Nv where N is the density of a given species, and v is the characteristic velocity. Prior to collision each particle travels the distance of the order of the mean free path $\lambda \sim (N_g\sigma)^{-1}$ (here σ is the characteristic cross section of collisions between the given species and other gas particles, and N_g is the total gas density). Hence, the net diffusion flux density of the particles is ΔNv where ΔN is the difference in the densities of the given species at two points separated by the distance of the order of the mean free path. And so we obtain $\Delta N \sim \lambda \operatorname{grad} N$ and the diffusion flux density $j \sim \lambda v \operatorname{grad} N$. A comparison with Eq. (7.1)* yields the following estimate for the diffusion coefficient**:

$$\mathscr{D} \sim v\lambda \sim \frac{T^{1/2}}{M^{1/2} N_g \sigma} \qquad (7.2)$$

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* We neglect here the sign of the flux density. Naturally, the direction is determined by the fact that it tends to equalize the particle density, so that the sign in Eq. (7.1) reflects this fact. The same can be said about the signs for other transport phenomena.

** This formula can be derived using dimensional analysis. The parameters T, M, and λ, which describe the motion of an individual particle in gas, make up only one combination with the dimensions of the diffusion coefficient $\mathscr{D}$, which is $\lambda T^{1/2}/M^{1/2}$.

Here T is the temperature of the gas, M is the mass of the particles, and it is assumed that the mass of the diffusing species are of the same order of magnitude as the mass of the other particles. Let us analyze the restoration of the equilibrium gas density after a density gradient has been established. Inserting (7.1) into the continuity equation (6.6), we obtain the equation for the particle density, which is known as the equation of diffusion:

$$\frac{\partial N}{\partial t} = \mathscr{D}\nabla^2 N \tag{7.3}$$

Let us denote by L the characteristic distance at which there occurs a noticeable variation of the gas density. Then Eq. (7.3) yields the characteristic time of the density variation, $\tau_L \sim L^2/\mathscr{D} \sim L^2/(v\lambda)$.

Macroscopic analysis of the transport of particles in gas is valid if the characteristic variations of the density occur in time intervals which are considerably larger than the characteristic time between two consecutive collisions of the particle $\tau \sim \lambda/v$. Hence, we obtain $\tau/\tau_L \sim (\lambda/L)^2 \ll 1$, that is, the macroscopic analysis of transport phenomena is valid if the gas density varies markedly at a distance considerably larger than the mean free path of the particles.

Energy and momentum transport in a gas. We have discussed the transport of particles in a gas. Transport of energy and momentum occurs in a similar way when the appropriate gradients exist in the gas. For instance, the temperature gradient gives rise to the heat flux vector defined as

$$\mathbf{q} = \int \mathbf{v}\,\frac{Mv^2}{2}\,f\,d\mathbf{v} \tag{7.4}$$

This heat flux vector tends to equalize the temperature of the gas in different regions; the flux vector is zero when the temperature is constant over the gas volume. Hence, the heat flux vector is proportional to the temperature gradient:

$$\mathbf{q} = -\varkappa \operatorname{grad} T \tag{7.5}$$

The factor $\varkappa$ is the thermal conductivity of the gas. In terms of our analysis, Eq. (7.5) is valid if the temperature of the gas varies only slightly over a distance of the order of the mean free path of the gas particles.

Let us estimate the magnitude of the thermal conductivity and its dependence on the system's parameters as it has been done for the diffusion coefficient (7.2). Imagine a plane perpendicular to the vector grad T and calculate the heat fluxes passing through it in both directions. According to Eq. (7.4), each of the fluxes is

$$q \sim Nv \times Mv^2 \sim NvT$$

where N is the density of particles, and v is their characteristic velocity.

These heat fluxes are due to the motion of the particles, which travel prior to collision a distance about the mean free path λ. Since the heat fluxes in both directions through the imaginary plane are due to the particles moving on different sides of the plane where the temperatures are different, the net flux is

$$q \sim Nv\Delta T$$

Here ΔT is the temperature difference over the distance of the order of the mean free path (the characteristic size of the region from which the particles get to the plane without collisions with other gas particles, that is, without energy exchange). Hence, we obtain $\Delta T \sim \lambda$ grad T and $q \sim \sim Nv\lambda$ grad T. A comparison with Eq. (7.5) yields the following estimate for the thermal conductivity:

$$\varkappa \sim Nv\lambda \sim \frac{v}{\sigma} \sim \frac{T^{1/2}}{M^{1/2}\sigma} \qquad (7.6)$$

where σ is the characteristic cross section for the colliding particles.

The thermal conductivity is seen to be independent of the density of the particles. Indeed, an increase in the particle density gives rise to a proportional increase in the number of particles transporting heat and to a proportional decrease in the mean free path of the particles, that is, the characteristic distance of heat transport. These two effects are mutually cancelled.

In the case of a directional motion of gas with the average velocity not being uniform over the cross section, the mean momentum is transported between the regions with different mean velocities of the gas. This gives rise to friction forces, which tend to slow the motion of gas with higher mean velocity and to equalize the mean velocity. The force F that acts (per unit area) opposite to the direction of the gas motion is

$$F=-\eta\frac{\partial w_x}{\partial z} \qquad (7.7)$$

Here w_x is the mean velocity of the gas (see Fig. 12), and the factor η is called the viscosity coefficient.

Let us estimate the magnitude of the viscosity coefficient and its dependence on the parameters of the system as it has been done for the diffusion coefficient and the thermal conductivity. The force (7.7) acting on the unit area is due to the exchange of particles with different mean velocities. Let us calculate this force.

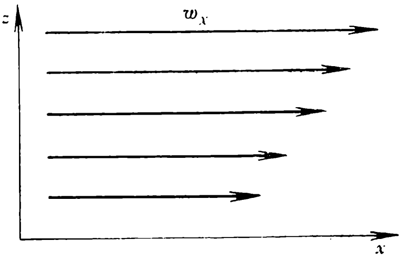
Variation of the mean momentum of the particles due to their exchange of position is $M\Delta w_x$ where M is the particle mass, and Δw_x is the difference between the mean velocities of directional motion. The flux density of particles is, by order of magnitude, equal to Nv where N is the density of particles, and v is their characteristic velocity. Therefore, the force acting on the unit area and equalling the variation of the mean momentum of particles per unit time is $F \sim NvM\Delta w_x$. Since the exchange of particles occurs between the regions separated by a distance of the order of the mean free path, we find that $\Delta w_x \sim \lambda\ \partial w_x/\partial z$ and $F \sim NvM\lambda\ \partial w_x/\partial z$. A comparison with Eq. (7.7) yields the following estimate for the viscosity coefficient:

$$\eta \sim NvM\lambda \sim \frac{T^{1/2}M^{1/2}}{\sigma} \qquad (7.8)$$

According to Eq. (7.8), the viscosity coefficient is independent of the density of particles. Similar to the thermal conductivity, this independence is due to compensation of two opposite effects occurring with momentum transport. Indeed, the number of momentum carriers is proportional to the density while the characteristic distance of transport, the mean free path of the particles, is inversely proportional to the gas density.

We have discussed the simplest transport phenomena caused by gradients of the macroscopic properties: the transport of particles caused by the density gradient, the heat transport caused by the temperature gradient, and the momentum transport caused by the gradient of the directional velocity. There can also occur transport phenomena of mixed origin. For instance, the density gradient can give rise to the heat transport, the temperature gradient can give rise to the transport of particles, and so on. These effects are due to specific relationships between the macroscopic parameters N, T, w, etc., which characterize the system. The picture can be even more complicated in the multicomponent system where a pressure gradient can be maintained for a long time for a given species. To analyze all these effects, we can use the general approach applied for deriving Eqs. (7.2), (7.6), and (7.8).

FIG. 12.
The mean velocity of the gas particles in the presence of viscosity.



Thermal conductivity due to the internal degrees of freedom. The characteristic cross section for the elastic collisions of particles is considerably larger than the cross section for inelastic collisions. Hence, the cross sections in Eqs. (7.2), (7.6), and (7.8) are those for the elastic collisions of particles while the inelastic collisions do not affect transport phenomena. One exception is the thermal conductivity of gas where the internal degrees of freedom affect the heat transport in a peculiar way. Let us assume that the gas at each point is in thermodynamic equilibrium with respect to the internal degrees of freedom. Then transport of particles between regions with different temperatures results in the transport of the internal energy of the particles though the internal energy difference is converted into the energy of translational motion (or vice versa) in the course of numerous elastic collisions. This phenomenon results in an increase in the thermal conductivity.

Let us represent the heat flux vector as a sum of two terms:

$$\mathbf{q} = -\varkappa_{\text{transl.}} \operatorname{grad} T - \varkappa_{\text{intern.}} \operatorname{grad} T \qquad (7.9)$$

Here the first term is due to the transport of translational energy of particles, similar to Eq. (7.5), while the second term is due to the internal degrees of freedom. Hence, the thermal conductivity is a sum of two terms:

$$\varkappa = \varkappa_{\text{transl.}} + \varkappa_{\text{intern.}} \qquad (7.10)$$

and we have to find the second term.

Let us describe the internal state of the particle by the label i. Since the spatial density of particles with a given internal state i is not uniform because of the temperature gradient, there is a diffusion flux density of these particles

$$\mathbf{j}_i = -\mathscr{D}_i \operatorname{grad} N_i = -\mathscr{D}_i \operatorname{grad} T \frac{\partial N_i}{\partial T}$$

where $\mathscr{D}_i$ is the diffusion coefficient of the particles with a given i. Denote by $\mathscr{E}_i$ the excitation energy of the ith internal state of the particle. Then the heat flux vector due to the internal degrees of freedom is

$$\mathbf{q}_{\text{intern.}} = \sum_i \mathscr{E}_i \mathbf{j}_i = -\sum_i \mathscr{E}_i \mathscr{D}_i \frac{\partial N_i}{\partial T} \operatorname{grad} T$$

A comparison with Eq. (7.9) yields the following expression for the thermal conductivity for the heat transport due to internal degrees of freedom:

$$\varkappa_{\text{intern.}} = \sum_i \mathscr{E}_i \mathscr{D}_i \frac{\partial N_i}{\partial T} \qquad (7.11)$$

Equation (7.11) is simplified if the diffusion coefficient does not depend on the excitation of the internal degrees of freedom of the particle. This is the case, for instance, when the respective internal degrees of freedom correspond to the rotational or vibrational excitation of molecules. In this case we obtain

$$\varkappa_{\text{intern.}} = \mathscr{D} \sum_i \mathscr{E}_i \frac{\partial N_i}{\partial T} = \mathscr{D} \frac{\partial}{\partial T} \sum_i \mathscr{E}_i N_i$$

$$= \mathscr{D} \frac{\partial}{\partial T} (\overline{\mathscr{E}} N) = \mathscr{D} N c_V \qquad (7.12)$$

Here $N = \sum_i N_i$ is the total density of the gas particles, $\overline{\mathscr{E}} = N^{-1} \sum_i \mathscr{E}_i N_i$ is the mean excitation energy of the particles, and c_V is the specific heat per molecule corresponding to excitation of the internal degrees of freedom. For instance, if excitation of the internal degrees of freedom is sufficiently effective, in other words, if $c_V \sim 1$, then Eqs. (7.12) and (7.2) yield $\varkappa_{\text{intern.}} \sim T^{1/2}/(M^{1/2}\sigma)$, that is, $\varkappa_{\text{intern.}}$ has the same order of magnitude as the thermal conductivity (7.6) due to the translational motion of particles.

Let us consider another special case when the thermal conductivity due to internal degrees of freedom is significant, namely, the thermal conductivity of a monoatomic gas with a slight admixture of molecules. The density of molecules N_m is related to the density of atoms N_a through Eq. (2.20):

$$N_a^2/N_m = f(T) \exp(-D/T)$$

Here $f(T)$ is a power function of temperature, and D is the dissociation energy of the molecule. Since $D \gg T$, we obtain

$$\frac{\partial N_m}{\partial T} = \frac{D}{T^2} N_m$$

Inserting this relation into Eq. (7.11), we find that

$$\varkappa_{\text{intern.}} = \mathscr{D} \left(\frac{D}{T}\right)^2 N_m \qquad (7.13)$$

where $\mathscr{D}$ is the diffusion coefficient for the molecules in the monoatomic gas. Compare this parameter with the thermal conductivity $\varkappa_{\text{transl.}}$ (7.6) due to the heat transport in the translational motion of molecules. Equations (7.2), (7.6), and (7.13) yield

$$\frac{\varkappa_{\text{intern.}}}{\varkappa_{\text{transl.}}} \sim \left(\frac{D}{T}\right)^2 \frac{N_m}{N_a} \qquad (7.14)$$

In the case being considered, we have $N_m \ll N_a$. However, since the complete dissociation of the molecules occur at the temperatures which are low compared to the molecule dissociation energy ($T \ll D$), the ratio (7.14) can be about unity for comparatively low concentrations of the molecules in the gas.

The mobility of particles. The Einstein relation. When a particle travels in vacuum in a field of external forces, it is uniformly accelerated. If this particle travels in gas, the collisions with the particles of the gas give rise to a friction force whose action determines the mean velocity of directional motion of the particle. When the external force is small, that is, the mean velocity $\mathbf{w}$ of directional motion of the particle is small compared to the thermal velocity of the gas particles, the velocity $\mathbf{w}$ is proportional to the acting force $\mathbf{F}$:

$$\mathbf{w} = b\mathbf{F} \qquad (7.15)$$

The proportionality factor b is known as the mobility of the particle. In the small-force approximation, the mobility does not depend on the magnitude of the external force $\mathbf{F}$.

Let us consider the motion of the test particles in a gas under the action of an external field, the particles being in thermodynamic equilibrium with the gas. According to the Boltzmann formula (2.9), we have the following spatial distribution of the density of the test particles:

$$N = N_0 \exp(-U/T) \qquad (7.16)$$

where U is the potential of the external field, T is the gas temperature, and N_0 is the density of the test particles at the point where the potential U is assumed to be zero.

Since the density of the test particles is not uniform over the cross section of the gas, the gradient of density gives rise to a diffusion flux density

$$\mathbf{j} = -\mathscr{D}\,\text{grad}\,N = \frac{\mathscr{D}\,\text{grad}\,U}{T}N = -\frac{\mathscr{D}}{T}\mathbf{F}N$$

Here we used the Boltzmann formula (2.9) for the density of the test particles and the relation $\mathbf{F} = -\operatorname{grad} U$ for the external force. Under thermodynamic equilibrium, when there is no net directional motion of the test particles, the diffusion flow of the test particles is compensated by the flow of the test particles under the action of the external force, the flow being $\mathbf{w}N = b\mathbf{F}N$ according to Eq. (7.15). Since the net flow is zero, we obtain the following relationship between the diffusion coefficient and the mobility of the test particles:

$$b = \mathcal{D}/T \qquad (7.17)$$

Equation (7.17) is known as the Einstein relation, and it is valid for the small fields which do not disturb the thermodynamic equilibrium between the test particles and the gas particles.

Equation (7.2) yields the following estimate for the mobility:

$$b \sim (MT)^{-1/2} (N\sigma)^{-1} \qquad (7.18)$$

The Navier-Stokes equation. Let us derive the equations for transport of momentum and energy in a viscous gas. The equation for momentum transport can be derived from Eq. (6.15) by taking into account the viscosity of the gas. Confining ourselves to the one-component gas, we can use Eq. (6.8) as the basis for the equation of momentum transport where we should add to the pressure tensor a term accounting for the viscosity. Using the frame of reference of Fig. 12 and Eq. (7.7), we find that this term, proportional to the viscosity coefficient, is given by

$$\mathcal{P}'_{xz} = -\eta \frac{\partial w_x}{\partial z}$$

Under our conditions the only nonzero component of the velocity is along the x axis and the acceleration is along the z axis.

Assuming that under the above conditions the pressure tensor is determined only by viscosity and noting that this tensor is symmetrical, we can write it in the following general form:

$$\mathcal{P}'_{ik} = -\eta \left(\frac{\partial w_i}{\partial x_k} + \frac{\partial w_k}{\partial x_i} + a\delta_{ik} \frac{\partial w_l}{\partial x_l} \right) \qquad (7.19)$$

Here summation is assumed over the subscripts appearing two times. The factor a can be found in the following way. The forces of viscous friction in gas are due to the fact that adjacent gas layers move with different velocities. If the gas is decelerated as a whole, that is, if the derivative $\partial w_i/\partial x_i$ is made nonzero, this mechanism of friction disappears and the force due to the gas viscosity must vanish. Hence, the trace of the pressure tensor due to the viscosity must be zero, that is, $\sum\limits_i \mathcal{P}'_{ii} = 0$. Thus, we find $a = -2/3$, and the viscosity term of the pressure tensor can be written as

$$\mathcal{P}'_{ik} = -\eta\left(\frac{\partial w_i}{\partial x_k} + \frac{\partial w_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial w_l}{\partial x_l}\right) \tag{7.20}$$

Let us add to the viscosity term of the pressure tensor (7.20) the term (6.13) due to the gas pressure. Inserting this pressure tensor into Eq. (6.8), we obtain

$$\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{w}\nabla)\,\mathbf{w} = -\frac{\operatorname{grad} p}{\rho} + \frac{\eta}{\rho}\nabla^2\mathbf{w} + \frac{\eta}{3\rho}\operatorname{grad}\operatorname{div}\mathbf{w} + \frac{\mathbf{F}}{M} \tag{7.21}$$

Equation (7.21) describes momentum transport and is called the Navier-Stokes equation.

The equation of heat transport. Let us derive the heat transport equation for the moving gas in which there occurs heat transport due to heat conduction. Let us write the balance equation for the variation of the energy of the gas in an elementary volume V. Denote by $\bar{\varepsilon}$ the mean energy of a gas particle and by N the density of the gas (to simplify the treatment we consider a one-component gas). The variation of energy in the volume V is given by

$$\frac{\partial}{\partial t}\int\limits_V \bar{\varepsilon} N\, d\mathbf{r} = -\oint\limits_S \mathbf{q}\, d\mathbf{s}$$

Here $\mathbf{q}$ is the heat flux vector, S is the surface enclosing the volume V, and $d\mathbf{s}$ is the element of the surface. The minus accounts for the fact that the heat outflow through the surface around the volume decreases the energy contained in the volume.

Making use of the Gauss theorem, we can transform the above equation into

$$\int_V \left[\frac{\partial}{\partial t}(\bar{\varepsilon}N) + \operatorname{div} \mathbf{q}\right] d\mathbf{r} = 0$$

Since this integral is zero for an arbitrary volume V, the integrand must be zero:

$$\frac{\partial}{\partial t}(\bar{\varepsilon}N) + \operatorname{div} \mathbf{q} = 0 \qquad (7.22)$$

This equation of heat transport has the same form as the continuity equation (6.6) for transport of particles.

The heat flux vector in Eq. (7.22) can be written as a sum of two terms. The first term is the heat flux $\bar{\varepsilon}N\mathbf{w}$ due to the gas motion. The second term is the heat flux due to thermal conduction, which is $-\varkappa$ grad T according to Eq. (7.5) where $\varkappa$ is the thermal conductivity coefficient. Inserting these heat fluxes into Eq. (7.22), we obtain the following equation for heat transport:

$$\frac{\partial}{\partial t}(\bar{\varepsilon}N) + \operatorname{div}(\bar{\varepsilon}N\mathbf{w}) - \varkappa\nabla^2 T = 0$$

Subtraction from the above equation of the continuity equation (6.6) yields

$$N\frac{\partial\bar{\varepsilon}}{\partial t} + N\mathbf{w}\operatorname{grad}\bar{\varepsilon} = \varkappa\nabla^2 T \qquad (7.23)$$

Assume that the mean energy of the molecule, which is the sum of the internal energy and the translational energy, is proportional to the temperature. This is the case if the gas temperature is the same as the temperature of the internal degrees of freedom. Hence, we obtain $\bar{\varepsilon} = c_V T$ where c_V is the specific heat per molecule*. Thus, the equation of heat transport (7.23) can be transformed into

$$\frac{\partial T}{\partial t} + \mathbf{w}\operatorname{grad} T = \frac{\varkappa}{c_V N}\nabla^2 T \qquad (7.24)$$

---

* For a monoatomic gas $c_V = 3/2$; if the gas temperature of a binary molecular gas is identical to the rotational temperature and if the majority of molecules are in the ground vibrational state and in the rotational states with high quantum numbers, then for this gas $c_V = 5/2$.

The diffusion motion of particles. Let us study the diffusion motion of particles in gas. Denote by $W(\mathbf{r}, t)$ the probability that a test particle is at point $\mathbf{r}$ at time t. Assume that at the zero time the test particle is at the origin so that the function $W(\mathbf{r}, t)$ is spherically symmetric. The function $W(\mathbf{r}, t)$ is normalized thus:

$$\int_0^\infty W(\mathbf{r}, t)\, 4\pi r^2\, dr = 1 \tag{7.25}$$

This probability function satisfies the diffusion equation (7.3) for the test particle travelling in the gas:

$$\frac{\partial W}{\partial t} = \mathscr{D}\nabla^2 W \tag{7.26}$$

Since the function W is spherically symmetrical, Eq. (7.26) can be written in the following form:

$$\frac{\partial W}{\partial t} = \mathscr{D}\frac{1}{r}\frac{\partial^2}{\partial r^2}(rW)$$

We are interested in average characteristics. Let us multiply the above equation by $4\pi r^4 dr$ and integrate the result over r. The left-hand side of the equation yields

$$\int_0^\infty 4\pi r^4\, dr\, \frac{\partial W}{\partial t} = \frac{d}{dt}\int_0^\infty r^2 W \times 4\pi r^2\, dr = \frac{d\overline{r^2}}{dt}$$

where $\overline{r^2}$ is the mean square of the distance. Integrating twice by parts and using Eq. (7.25), we transform the right-hand side of the equation into

$$\mathscr{D}\int_0^\infty 4\pi r^4\, dr\, \frac{1}{r}\frac{\partial^2}{\partial r^2}(rW) = -3\mathscr{D}\int_0^\infty 4\pi r^2\, dr\, \frac{\partial (rW)}{\partial r}$$

$$= 6\mathscr{D}\int_0^\infty 4\pi r^2\, dr\, W = 6\mathscr{D}$$

The resulting equation is

$$\frac{d\overline{r^2}}{dt} = 6\mathscr{D}$$

Since at zero time the particle was at the origin, the solution of the equation is given by

$$\overline{r^2} = 6\mathscr{D}t \qquad (7.27)$$

In the case of spherical symmetry, we have $\overline{x^2} = \overline{y^2} = \overline{z^2} = \overline{r^2}/3$ so that

$$\overline{x^2} = \overline{y^2} = \overline{z^2} = 2\mathscr{D}t \qquad (7.28)$$

Equations (7.27) and (7.28) describe the diffusion motion of the particle characterized by numerous collisions with the gas particles so that each collision changes the direction of the particle's motion. This motion is sometimes called Brownian motion. For Brownian motion the distance travelled varies as the square root of time.

Convective instability of a gas. If the temperature gradient is large in a gas which is in a field of external forces, there may appear a more effective mechanism of heat transport than thermal conduction, which is called convection. This process consists in the movement of the warmer gas into the colder regions and the colder gas into the warmer regions.

Let us analyze the stability of a gas at rest with respect to development of convection. Consider a gas at rest in which a temperature gradient is maintained in the field of external forces. The parameters of the gas are subjected to a small perturbation, which is due to a slow motion of the gas and corresponds to convection. If this perturbation proves to be possible, convective instability can develop in such a gas. We have to find what conditions are necessary for the development of the convective instability, that is, for convective heat transport.

Let us formulate the simplest problem, namely, the so-called Rayleigh problem. The gas fills a gap between two infinite parallel walls. The temperature of the lower wall is T_1, the temperature of the upper wall is T_2, and T_1 is higher than T_2. The force of the external field is directed downwards perpendicular to the walls. The distance between the walls is L. Find the conditions for the development of convective instability.

Let us represent the parameters of the gas as sums of two terms: the first term is the parameter for the resting gas and the second term is a small perturbation of the parameter due to the convective motion of the gas. Thus, the gas density is $N+N'$, the gas pressure is p_0+p', the gas temperature is $T+T'$, and the gas velocity is $\mathbf{w}$ (it is zero in the absence of convection). Insert these parameters into the stationary equations of continuity (6.6), of momentum transport (7.21), and of heat transport (7.24). The zero-order approximation is

$$\operatorname{grad} p_0 = -\mathbf{F}N, \quad \nabla^2 T = 0$$

In the first small-parameter approximation these equations yield

$$\left.\begin{aligned} \operatorname{div}\mathbf{w}=0,\\ -\frac{\operatorname{grad}(p_0+p')}{N+N'}+\frac{\eta\nabla^2\mathbf{w}}{N+N'}+\mathbf{F}=0,\\ w_z\frac{(T_2-T_1)}{L}=\frac{\varkappa}{c_V N}\nabla^2 T' \end{aligned}\right\} \qquad (7.29)$$

The parameters of the above problem are used in the last equation. Here the z axis is perpendicular to the walls.

Transform the first term in the second equation (7.29). Up to the first order of approximation, this term is

$$\frac{\operatorname{grad}(p_0+p')}{N+N'}=\frac{\operatorname{grad}p_0}{N}+\frac{\operatorname{grad}p'}{N}-\frac{\operatorname{grad}p_0}{N}\frac{N'}{N}$$
$$=\mathbf{F}\left(1-\frac{N'}{N}\right)+\frac{\operatorname{grad}p'}{N}$$

According to the equation of state (6.14), for a gas $N=p/T$, and so we find that $N'=(\partial N/\partial T)_p T'=-NT'/T$. Inserting this relation into the second equation, we can write the system (7.29) in the following form:

$$\left.\begin{aligned} \operatorname{div}\mathbf{w}=0,\\ \frac{\operatorname{grad}p'}{N}-\mathbf{F}\frac{T'}{T}-\frac{\eta}{N}\nabla^2\mathbf{w}=0,\\ w_z=\frac{\varkappa L}{c_V N(T_2-T_1)}\nabla^2 T' \end{aligned}\right\} \qquad (7.30)$$

Let us reduce the system of equations (7.30), which connect the parameters of the gas, to an equation for one parameter. First, we apply to the second equation of (7.30) the operator div and take into account the first equation of (7.30). We find that

$$\frac{\nabla^2 p'}{N} - \frac{F}{T}\frac{\partial T'}{\partial z} = 0 \qquad (7.31)$$

Here we assume that $(T_1 - T_2)/T_1 \ll 1$. Therefore, the unperturbed parameters of the gas do not vary much inside the volume being considered; we shall neglect their variation and assume that the unperturbed gas parameters are spatially constant.

Inserting w_z from the third equation of (7.30) into the zth component of the second equation and applying the operator ∇^2 to the result, we obtain the following equation:

$$\frac{1}{N}\frac{\partial}{\partial z}\nabla^2 p' - F\frac{\nabla^2 T'}{T} + \frac{\eta\varkappa L}{c_V N^2 (T_2 - T_1)}(\nabla^2)^3 T' = 0$$

Using the relation (7.31) between $\nabla^2 p'$ and T', we obtain finally

$$(\nabla^2)^3 T' = -\frac{R}{L^4}\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right) T' \qquad (7.32)$$

where the dimensionless combination of parameters

$$R = \frac{(T_1 - T_2)\, c_V F N^2 L^3}{\eta\varkappa T} \qquad (7.33)$$

is called the Rayleigh number.

Equation (7.32) shows that the Rayleigh number determines the possibility of the development of convection. For instance, in the Rayleigh problem the boundary conditions at the walls are $T' = 0$, $w_z = 0$. Also, the tangential forces $\eta(\partial w_x/\partial z)$ and $\eta(\partial w_y/\partial z)$ are zero at the walls. Differentiating the equation $\operatorname{div} \mathbf{w} = 0$ with respect to z and using the conditions for the tangential forces, we find that at the walls $(\partial^2 w_z/\partial z^2) = 0$. Hence, we have the following boundary conditions:

$$T' = 0, \quad w_z = 0, \quad \frac{\partial^2 w_z}{\partial z^2} = 0$$

Denote by $z = 0$ the coordinate of the lower wall and by $z = L$ the coordinate of the upper wall. The general solution of Eq. (7.32) with the boundary condition $T' = 0$ at

$z = 0$ can be expressed as

$$T' = C \exp [i (k_x x + k_y y)] \sin k_z z \qquad (7.34)$$

The boundary condition $T' = 0$ at $z = L$ yields $k_z L = \pi n$ where n is an integer. Inserting the solution (7.34) into Eq. (7.32), we obtain

$$R = \frac{(k^2L^2 + \pi^2 n^2)^3}{k^2 L^2} \qquad (7.35)$$

where $k^2 = k_x^2 + k_y^2$. The solution (7.34) satisfies all boundary conditions.

Equation (7.35) shows that convection can occur for Rayleigh numbers not less than a minimum number $R_{\min}$ corresponding to $n = 1$ and $k_{\min} = \pi/(L\sqrt{2})$:

$$R_{\min} = \frac{27\pi^4}{4}$$

The magnitude of $R_{\min}$ varies according to the geometry of the problem, but in all problems the Rayleigh number characterizes the possibility of convection.

Convective motion of a gas. Let us analyze the simple case of motion of gas in the plane xz. Inserting the solution (7.34) into the equation

$$\operatorname{div} \mathbf{w} = \frac{\partial w_x}{\partial x} + \frac{\partial w_z}{\partial z} = 0$$

we find the following expressions for the components of the gas velocity:

$$\left. \begin{aligned} w_z &= w_0 \cos kx \sin \frac{\pi n z}{L}, \\ w_x &= -\frac{\pi n}{kL} w_0 \sin kx \cos \frac{\pi n z}{L} \end{aligned} \right\} \qquad (7.36)$$

where n is an integer, and the characteristic velocity of gas w_0 is assumed to be small compared to the respective parameters in the gas at rest; for instance, w_0 is small compared to the thermal velocity of the particles.

Let us trace the motion of a gas element. The equations of motion for this element are $dx/dt = w_x$ and $dz/dt = w_z$; using Eqs. (7.36) for the components of the gas velocity, we obtain

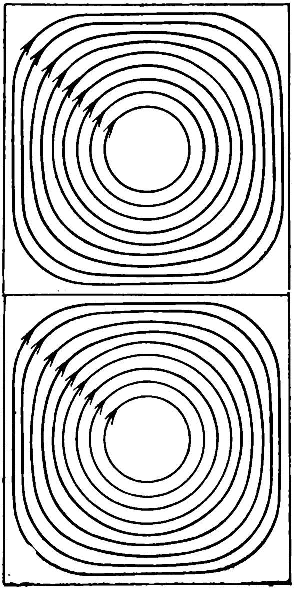
$$\frac{dx}{dz} = -\frac{\pi n}{kL} \tan kx \cot \frac{\pi n z}{L}$$

This equation describes the path of the gas element. The equation yields

$$\sin kx \sin \frac{\pi n z}{L} = C \qquad (7.37)$$

where C is a constant determined by the initial conditions.

FIG. 13.
The paths of the gas elements in the Rayleigh problem for $n=1$, $k=\pi/L$.



The constant C varies from -1 to $+1$; its value depends on the initial positions of the gas elements being considered. Of special significance are the lines at which $C = 0$. These lines are given by the following equations:

$$z = \frac{L}{n} m_1, \quad x = \frac{\pi}{k} m_2 \qquad (7.38)$$

where m_1 and m_2 are integers, and $m_1 \geqslant 0$ and $m_2 \geqslant 0$. The straight lines (7.38) divide the gas into cells. The gas particles which are inside a cell can travel only inside this cell and cannot leave it. Indeed, Eqs. (7.36) show that the component of the gas velocity normal to the cell boundary is zero, that is, the gas cannot cross the boundary between the cells. These cells are known as the Benard cells.

Equation (7.37) shows that inside each cell the gas elements travel along closed paths around the centre where the gas is resting. Figure 13 shows the paths of the gas elements in the Rayleigh problem for $n = 1$ and $k = \pi/L$, that is, for the Rayleigh number $R = 8\pi^4$. In the Rayleigh problem, the Benard cells are pyramids with regular polygon bases; in the general case, these cells have a more complicated shape.

Convective heat transport. Compared to the thermal conduction, convection is a more effective mechanism of heat transport. Let us analyze heat transport in the Rayleigh problem. The resting boundary layers of thickness δ are formed at the walls whose temperatures are constant. The thickness of the boundary layer is determined by the viscosity of the gas, and the heat transport on the boundary layer is due to thermal conduction. Then this heat is transported by gas flows so that the heat flux transported in the Rayleigh problem equals the heat flux due to thermal conduction in the boundary layer:

$$q = \varkappa \, |\operatorname{grad} T| \sim \frac{\varkappa (T_1 - T_2)}{\delta}$$

Applying the Navier-Stokes equation (7.21) to the boundary region, we can estimate the thickness of the boundary layer. This equation describes continuous transition between the wall and the moving gas flows. Now to the second equation of (7.30) we have to add the term $M(\mathbf{w}\nabla)\mathbf{w}$, which cannot be neglected here. A comparison of separate terms of the zth component of the resulting equation yields, by order of magnitude,

$$M\frac{w_z^2}{\delta} \sim \frac{F(T_1 - T_2)}{T} \sim \frac{\eta}{N}\frac{w_z}{\delta^2} \qquad (7.39)$$

Hence, we obtain

$$\delta \sim \left[\frac{\eta^2 T}{N^2 F M (T_1 - T_2)}\right]^{1/3} \qquad (7.40)$$

Let us compare the heat flux transported by gas in the above problem and the heat flux transported due to heat conduction in the resting gas. The heat flux due to heat conduction is $q_{\text{cond.}} \sim \varkappa (T_1 - T_2)/L$ where L is the separation of the walls; thus, we have

$$\frac{q}{q_{\text{cond.}}} \sim \frac{L}{\delta} \sim \left[\frac{N^2FML^3(T_1-T_2)}{\eta^2 T}\right]^{1/3} \sim G^{1/3}\,. \qquad (7.41)$$

Here G is the dimensionless combination of the parameters,

$$G = \frac{N^2FM(T_1-T_2)L^3}{\eta^2 T} \qquad (7.42)$$

known as the Grashof number. A comparison of the Rayleigh number (7.33) and the Grashof number (7.42) yields the ratio between them

$$\frac{R}{G} = \frac{c_V \eta}{M\varkappa}$$

Note that the continuity equation (6.6), the equation of momentum transport (7.21), and the equation of heat transport (7.24) used above are valid not only for a gas but also for a liquid. Therefore, the results obtained are applicable to liquids as well. However, gas has some peculiarities in this respect. For instance, Eqs. (7.6) and (7.8) yield that for gas the ratio $\eta/(M\varkappa)$ is about 1. Furthermore, the specific heat c_V of one molecule is also about 1. Hence, the Rayleigh number has the same order of magnitude as the Grashof number for gas. Since convection develops at high Rayleigh numbers, we find that for convection $G \gg 1$. Hence, according to Eq. (7.41), we find that heat transport via convection is considerably more effective than heat transport in a resting gas via heat conduction.

The ratio (7.41) between the heat fluxes due to convection and thermal conduction has been derived for an external force directed perpendicular to the boundary layer. Let us derive the respective relation for the external force directed parallel to the boundary layer. Let the z axis be normal to the boundary layer and the external force be directed along the x axis. Then to the second equation of the system (7.30) we add the term $M(\mathbf{w}\nabla)\mathbf{w}$ and compare the terms of the xth component of this equation. The result is, instead of Eq. (7.39), equal to

$$M\frac{w_x^2}{L} \sim F\frac{(T_1-T_2)}{T} \sim \frac{\eta}{N}\frac{w_x}{\delta^2} \tag{7.43}$$

Equation (7.43) yields an estimate of the thickness of the boundary layer:

$$\delta \sim \left[\frac{\eta^2 TL}{N^2 FM(T_1-T_2)}\right]^{1/4}$$

Hence, we find the ratio between the heat fluxes of convection and thermal conduction:

$$\frac{q}{q_{\text{cond.}}} \sim \frac{L}{\delta} \sim G^{1/4} \tag{7.44}$$

The ratio between the heat fluxes in this case is seen to be different from that for the external force perpendicular to the boundary layer [see Eq. (7.41)]. However, the convective heat flux in this case is still considerably larger than the heat flux due to the thermal conduction in the resting gas.

The instability of convective motion. New types of convective motion develop when the Rayleigh and Grashof numbers increase. This disturbs the ordered convective flow and finally disrupts the stability of the convective motion of gas giving rise to disordered or turbulent flow of gas even if it is contained in a resting closed system. Let us analyze the development of turbulent gas flow.

We consider once again the Rayleigh problem: a gas at rest is in an external field between two infinite and parallel planes maintained at different constant temperatures. We analyze the convective motion of gas (7.36) corresponding to sufficiently high Rayleigh numbers with $n \geqslant 2$. In this case there can simultaneously develop at least two different types of convective motion. Figure 14 shows two types of convective motion for the Rayleigh number $R = 108\pi^4$ corresponding to the wave number $k_1 = 9.4/L$ for $n = 1$, and $k_2 = 4.7/L$ for $n = 2$.

Let us analyze the example of convection illustrated in Fig. 14. Using the above parameters, we combine the solutions so that the gas flows corresponding to $n = 1$ and $n = 2$ travel in the same direction in some regions of the gas volume. Then in other regions these flows will necessarily travel in different directions. Two types of motion can develop for any Rayleigh numbers.

The fact that the problem has two solutions corresponding to opposite gas flows in some regions does not necessarily mean that the ordered flow of gas is disturbed. Indeed, we can always make a combination of two solutions describing a mixture of the two types of motion. For instance, if two gas flows travel in the opposite directions with the same velocity and amplitude, such a combination of solutions corresponds to a gas that is stationary in the given volume.

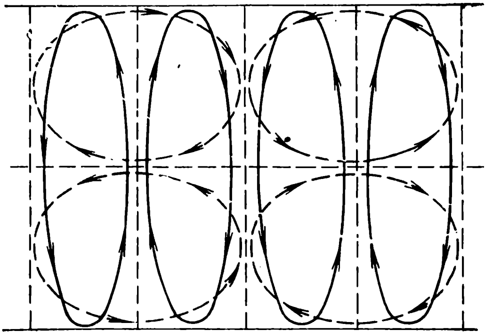


FIG. 14. The types of convective motion in the Rayleigh problem for $R=108\pi^4$. The mixing of the gas flows travelling in opposite directions finally results in a random gas motion, or turbulence.

Nevertheless, the fact that increasing the Rayleigh number gives rise to new types of solutions means that the convective flow can become turbulent. We assume that in the system there is ordered convective flow corresponding to one of the solutions. Then a small perturbation in one of the regions of the gas volume gives rise to another type of flow. At the boundary of this region two opposite gas flows meet so that the kinetic energy of motion of the gas flows transforms into the thermal energy of gas. This results in a disordered motion of the gas. The development of turbulence changes the character of heat transport.

8 Transport of Charged Particles in Weakly Ionized Gas

The mobility of charged particles. The conductivity of a weakly ionized gas. Let us analyze the motion of charged particles in a gas in the presence of an external electric field. The mobility of charged particles K is defined as the proportionality factor between the mean velocity of charged particles $\mathbf{w}$ and the electric field strength $\mathbf{E}$:

$$\mathbf{w} = K\mathbf{E} \tag{8.1}$$

The force of the electric field acting on the electron or singly charged ion is $\mathbf{F} = e\mathbf{E}$, and a comparison of Eqs. (8.1) and (7.15) yields

$$b = K/e \tag{8.2}$$

Hence, for charged particles the Einstein relation (7.17) is transformed into

$$K = e\mathcal{D}/T \tag{8.3}$$

where $\mathcal{D}$ is the diffusion coefficient of the charged particle in the gas. Equations (8.3), (7.2), and (7.18) yield the following estimate for the mobility of a charged particle (an electron or a singly charged ion) in gas:

$$K \sim e(MT)^{-1/2}(N\sigma)^{-1} \tag{8.4}$$

where N is the gas density, and σ is the characteristic cross section of collision between the charged particle and the gas particle.

Let us find the conductivity Σ of a weakly ionized gas, that is, of a gas which has a small concentration of charged particles, that is, a small ratio between the current of the charged particles and the electric field strength. The electric current due to the electrons is

$$ewN_e = eK_eN_eE$$

where N_e is the density of electrons, K_e is the mobility of electrons, and w is the mean velocity of electrons in the electric field E. A similar expression can be written for the electric current due to the ions. Equation (8.4) shows that the mobility is inversely proportional to the square root of the particle's mass. Since the densities of electrons and ions in the quasineutral plasma, which is the one we usually consider, are the same, the electric current in the weakly ionized gas is due mostly to electrons, and the gas conductivity is

$$\Sigma = eK_eN_e \tag{8.5}$$

Using Eqs. (8.4) and (8.5), we can estimate the conductivity of a weakly ionized gas:

$$\Sigma \sim N_e e^2 (mT)^{-1/2} (N\sigma_{ea})^{-1} \tag{8.6}$$

Here m is the electron mass, and σ_{ea} is the characteristic cross section for collisions between electrons and the neutral gas particles whose density is N.

Ambipolar diffusion. Assume that a plasma has been formed in a small volume of gas and it tends to spread over the whole gas volume. Let us analyze the decay of this plasma. In our case, the charged particles are spreading in the gas volume via the diffusion mechanism. Equation (7.2) shows that the electrons have a considerably larger diffusion coefficient than the ions; therefore, the electrons are spreading over the gas volume considerably faster than the ions. This results in a disturbance of quasineutrality of the plasma and in the emergence of electric fields in the plasma created by the charged particles.

The electric field **E** created due to the spatial distribution of the charged particles satisfies Poisson's equation:

$$\operatorname{div} \mathbf{E} = 4\pi e (N_i - N_e) \tag{8.7}$$

where N_e and N_i are the densities of the electrons and ions*. The flux densities of the electrons and ions in the system are the sums of the diffusion flux density and the flux density due to the electric field:

$$\mathbf{j}_e = -\mathscr{D}_e \operatorname{grad} N_e - K_eN_e\mathbf{E}, \tag{8.8a}$$

$$\mathbf{j}_i = -\mathscr{D}_i \operatorname{grad} N_i - K_iN_i\mathbf{E} \tag{8.8b}$$

In the above equations $\mathscr{D}_e$ and $\mathscr{D}_i$ are the diffusion coefficients of the electrons and ions in the gas, and K_e and K_i are the mobilities of the electrons and ions.

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* From now on the ions are assumed to be singly charged.

Let us consider such a mode of development that the plasma remains quasineutral in the process of motion. This is the case for relatively high densities of charged particles; the separation of charges gives rise to large fields which prevent further separation and preserve the quasineutrality of the plasma. This phenomenon is termed the ambipolar diffusion. In this case $N_e = N_i = N$ so that $\Delta N \equiv |N_e - N_i| \ll N$, and the fluxes of the electrons and the ions are the same.

Let us analyze Eqs. (8.8) for the electron and ion flux densities in the case of the ambipolar diffusion. Equation (7.2) for the diffusion coefficient and Eq. (8.4) for the mobility show that each of the terms in Eq. (8.8a) for the electron flux is much larger than the respective term in Eq. (8.8b) for the ion flux. For the electron flux to be equal to the ion flux, the first term in Eq. (8.8a) must almost cancel the second term. This means that the electric field strength must be $\mathbf{E} = -(\mathscr{D}_e/K_e) \times (\text{grad } N)/N = -(T_e/e) \times$ $\times (\text{grad } N)/N$. Inserting this into Eq. (8.8b), we find the flux density of the charged particles:

$$\mathbf{j} = -\mathscr{D}_a \text{ grad } N \qquad (8.9)$$

where

$$\mathscr{D}_a = \mathscr{D}_i (1 + T_e/T_i) \qquad (8.10)$$

is known as the ambipolar diffusion coefficient. In the above analysis, we made use of the Einstein relation (7.17) and assumed that the electron temperature T_e differs from the ion temperature T_i. Thus, the ambipolar diffusion of the charged particles is a diffusion-like motion with the time parameter corresponding to the ions.

Let us determine the conditions under which the ambipolar diffusion occurs. Assume that the characteristic size of the plasma at which the plasma parameters vary noticeably is L. The above expression for the electric field strength yields the following estimate (grad $N \sim N/L$):

$$E \sim T_e/eL$$

Poisson's equation yields

$$\Delta N \equiv |N_e - N_i| = \frac{|\text{div } \mathbf{E}|}{4\pi e} \sim \frac{T_e}{4\pi e^2 L^2}$$

Using the Debye-Hückel radius (3.6), we find that

$$\frac{\Delta N}{N} \sim \left(\frac{r_D}{L}\right)^2$$

With the above condition for ambipolar diffusion, $\Delta N \ll N$, we obtain the following condition for the realization of ambipolar diffusion:

$$r_D \ll L \qquad (8.11)$$

that is, the Debye-Hückel radius of the plasma must be considerably smaller than the characteristic size.

The mobility of ions in a foreign gas. Equation (8.4) gives an estimate of the mobility of ions for small electric fields. Now we shall analyze ion motion in gas under real conditions for arbitrary electric fields. First, consider the motion of ions in a foreign gas where the scattering of the ion is due to elastic collisions between the ion and the gas particles. For low collision energies, the cross section of the collisions is determined by the polarization interaction between the ion and the gas particle: $U = -\beta e^2/(2R^4)$ (here β is the polarizability of the gas particles, e is the ion or electron charge, and R is the distance between the ion and the particle). Equation (4.11) then gives the cross section for the capture of an ion by an atom:

$$\sigma_{\text{capture}} = 2\pi \left(\frac{\beta e^2}{2\varepsilon}\right)^{1/2} \qquad (8.12)$$

Here $\varepsilon = \mu v^2/2$ is the energy of the relative motion of the particles, μ is the reduced mass, and v is the relative velocity of the collision. The frequency of collisions between the ions and the gas particles (atoms) does not depend on the velocity of collision.

Taking into account that the mean energy of ions, ε, is of the order of the thermal energy, we find from Eqs. (8.4) and (8.12) the mobility of the ions as a function of the parameters of the problem:

$$K \sim \frac{1}{N(\beta\mu)^{1/2}} \qquad (8.13)$$

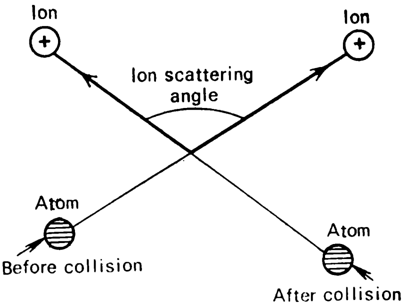
The mobility of ions in the parent gas. When ions travel in the parent gas, their mobility is determined by the resonance charge exchange between ions and gas particles.

This is the case when an ion can be produced from the gas particle (atom or molecule) by removing one electron. The resonance charge exchange can be described by the following scheme:

$$\tilde{A}^{+} + A \rightarrow \tilde{A} + A^{+} \qquad (8.14)$$

Here the tilde designates one of the nuclei. The process consists in the electron transferring from one atomic core to another.

FIG. 15.
The relay process of resonance charge exchange between an ion and an atom.



For thermal collision energies, the cross section of the resonance charge exchange is considerably larger than the cross section of the elastic collision between the ion and the atom. Therefore, the main contribution to the cross section of the resonance charge exchange is made by the collisions in which the atom and the ion travel along straight lines. Under these circumstances the ion travels in the gas via a relay mechanism (see Fig. 15). The resonance charge exchange changes the path of the ion since it is associated now with another nucleus. Therefore, into Eq. (8.4) for the ion mobility for low electric fields we must insert the cross section of the resonance charge exchange instead of the cross section of the elastic collision between the ion and the gas particle.

Let us analyze the ion mobility in the parent gas for high electric fields when the mean ion energy is considerably higher than the thermal energy of the gas particles. Under such conditions the gas particles colliding with the ion can be assumed to be at rest. Hence, we can visualize the motion of the ion in the following way. The resonance charge exchange between the ion and the gas particle gives rise to an ion with a zero velocity, the ion then being accelerated by the electric field until the next charge exchange between it and a gas particle takes place. Therefore, the only nonzero velocity component of the ion, $v_x = (eE/M_i)t$, is along the electric field (M_i is the ion mass, and t is the time elapsed since the last charge exchange).

Denote by $W(t)$ the probability that the ion will not make a charge exchange with a gas particle during the time t after the last charge exchange. This probability function satisfies the equation of radioactive decay:

$$\frac{dW}{dt} = -\nu W$$

here $\nu = N v_x \sigma_{\text{res.}}$ is the frequency of the resonance charge exchange, N is the density of the gas particles, and $\sigma_{\text{res.}}$ is the cross section of the resonance charge exchange. The solution of this equation is

$$W(t) = \exp\left[-\int_0^t \nu(t')\,dt'\right]$$

The velocity distribution of ions is described by the function $f(v_x)$. Since the ion velocity v_x along the field is determined by the time t elapsed since the last charge exchange, the distribution function $f(v_x)$ is proportional to the probability that the ion has not exchanged charge up to the moment t. Using the normalization condition, we find the distribution function

$$f(v_x) = N_i\left(\frac{2M_i}{\pi eE\lambda}\right)^{1/2} \exp\left(-\frac{M_i v_x^2}{2eE\lambda}\right), \quad v_x > 0 \qquad (8.15)$$

Here N_i is the ion density, $\lambda = 1/(N\sigma_{\text{res.}})$ is the mean free path of the ions, and the cross section of the resonance charge exchange is assumed independent of the velocity, as is the case in reality. Hence, we obtain the following expression for the mean velocity and the mean energy of the ion:

$$w = \langle v_x \rangle = \left(\frac{2eE\lambda}{\pi M_i}\right)^{1/2} \qquad (8.16)$$

$$\left\langle \frac{1}{2} M_i v_x^2 \right\rangle = \frac{1}{2} eE\lambda \qquad (8.17)$$

It may be seen that for high electric fields the ion mobility K varies with the field strength as $E^{-1/2}$. The above equations hold when the characteristic ion energy is considerably higher than the thermal energy:

$$eE\lambda \gg T \qquad (8.18)$$

Equation (8.18) is the criterion for a high field in terms of this problem. When the condition (8.18) is not satisfied, the drift velocity of the ion is small compared to the thermal velocity, the velocity distribution function for the ions is close to the Maxwell distribution, and the ion mobility does not depend on the electric field strength for any types of collisions between the ion and the gas particle. Under these circumstances the electric field is assumed to be small. When the condition (8.18) is satisfied, these relations do not hold.

Recombination of ions in a dense gas. Let us analyze recombination of the positive and negative ions in high-density gas for which the conditions opposite to conditions (5.14) and (5.21) are satisfied. Under such circumstances, frequent collisions of the ions with the gas particles prevent them from approaching each other and thus the time of recombination is, primarily, the time required for approaching. If the separation of the ions is R, each ion is under the action of the field produced by another ion whose strength is e/R^2. This field makes the ions move towards each other with a velocity $w = e(K_+ + K_-)/R^2$ where K_+ and K_- are the mobilities of the positive and negative ions in the gas. The expression for the drift velocity w is valid for $R \gg \lambda$ where λ is the mean free path of the ions in the gas.

Now let us determine the frequency of decay of the positive ions due to recombination. Imagine a sphere of radius R around the positive ion and compute the number of negative ions entering this sphere per unit time. The number is the sphere surface area times the flux density of the negative ions $N_-w = N_-e(K_+ + K_-)/R^2$. Proceeding from this, we write the balance equation for the density N_+ of the positive ions as

$$\frac{dN_+}{dt} = -N_+N_-4\pi e(K_+ + K_-)$$

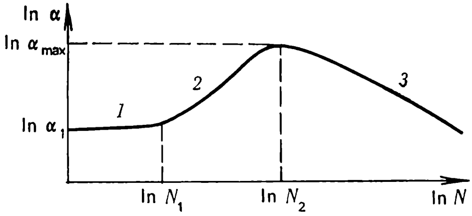
A comparison with Eq. (5.1) yields the recombination coefficient

$$\alpha = 4\pi e (K_+ + K_-) \tag{8.19}$$

Equation (8.19) is known as the Langevin formula.

Let us consider this type of recombination in more detail. The ions approach under the action of the Coulomb force and collide with the gas particles in the course of this approach. When the distance between the ions is R, we have $e^2\lambda/R^2 \ll T$; this relation is the opposite of the condition (8.18). Hence, the mean kinetic energy of the ions is the same as for an infinite distance between the nuclei. This is the case also for $R \sim b$ [see Eq. (5.12)], when the potential of the interaction between the ions is of the order or more than their thermal energy, that is, the ions are in a bound state. Thus, the main difficulty encountered in this type of ion recombination is not the production of the bound state as is the case for the triple recombination but the approach of the ions up to the distances at which charge exchange can occur. The higher the gas density the more difficult is the approach, and Eqs. (8.4) and (8.19) indicate that the recombination coefficient for ions in a dense gas decreases with increasing gas density.

FIG. 16.
The recombination coefficient of positive and negative ions as a function of gas density.



The recombination coefficient of ions as a function of gas density. We have determined the recombination coefficient of the positive and negative ions in a gas under various conditions. Thus we have the general picture of the ion-ion recombination. Figure 16 presents the qualitative relationship between the recombination coefficient and the gas density. Let us analyze this relationship.

For low gas densities (region *1*) recombination is due to the pair collisions of ions, the recombination coefficient is given by Eq. (5.9), and the order-of-magnitude estimate of the coefficient is $\alpha_1 \geqslant \hbar^2/(m^2\mu T)^{1/2}$ since in Eq. (5.9) we have $R_0 > a_0$. Here μ is the reduced mass of the ions, m is the electron mass, and $a_0 = \hbar^2/(me^2)$ is the Bohr radius. The recombination coefficient in region *2* is given by Eq. (5.20); we have $\alpha_2 \sim [C]\,(e^6/T^3)\,(\beta e^2/M)^{1/2}$ where β is the polarizability of the particle species C, $[C]$ is its density and M is the mass of particle C.

The gas densities of the order of $N_1 \sim a_0\,(T/e^2)^{5/2}\beta^{-1/2}$ correspond to the transition from region *1* to region *2*.

Region *3* is described by the Langevin theory and Eqs. (8.13) and (8.19) give the following order-of-magnitude estimate for the recombination coefficient in this field: $\alpha_3 \sim e\,[C]/(\beta\mu)^{1/2}$. The transition between regions *2* and *3* corresponds to the density $N_2 \sim (T/e^2)^{3/2}\beta^{-1/2}$. This density corresponds to the greatest recombination coefficient $\alpha_{\max} \sim \sim e^4\mu^{-1/2}T^{-3/2}$; it is found for the pair process assuming the Coulomb interaction cross section. Thus, the greatest recombination coefficient has the same order of magnitude as the rate constant for the elastic collision of ions.

Given below are the numerical values of the qualities indicated in Fig. 16 for the ions produced in air at room temperature: $N_1 \sim 10^{17}$ cm^{-3}, $N_2 \sim 10^{20}$ cm^{-3}, $\alpha_1 \sim \sim 10^{-9}$ cm^{3}s^{-1}, and $\alpha_{\max} \sim 10^{-6}$ cm^{3}s^{-1}.

9 Plasma in External Fields

The electron motion in a gas in an external field. Let us consider the motion of electron gas to which electric and magnetic fields are applied. To make a correct qualitative description of the relevant phenomena let us make use of the simplest form of the collision integral, that is, the tau approximation (6.4). The kinetic equation for the electron distribution function can be written as

$$\frac{\partial f}{\partial t} + \frac{\mathbf{F}}{m}\frac{\partial f}{\partial \mathbf{v}} = -\frac{f - f_0}{\tau} \qquad (9.1)$$

where $\mathbf{F}$ is the force of the external field acting on the electron, m is the electron mass, f_0 is the equilibrium distribution function realized in the absence of external fields, $1/\tau = Nv\sigma_{ea}$ is the frequency of the elastic collisions of electrons and the gas particles, N is the gas density, v is the electron velocity, and σ_{ea} is the cross section of elastic collisions of electron and the gas particles. To simplify the analysis we shall assume that τ is independent of the electron velocity.

Let us study the force $\mathbf{F}$ of the external magnetic and electric fields acting on the electron. We assume that the electric field varies harmonically and the magnetic field is constant and normal to the electric field. From this general case we can derive all the special cases of interest for us. The force which acts on the electron is

$$\mathbf{F} = e\mathbf{E}\exp(i\omega t) + \frac{e}{c}\,\mathbf{v}\times\mathbf{H} \qquad (9.2)$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, ω is the frequency of variation of the electric field, and $\mathbf{v}$ is the electron velocity. Let the vector $\mathbf{H}$ be parallel to the z axis and the vector $\mathbf{E}$ lie in the xy plane, that is, $\mathbf{H} = \mathbf{k}H$ and $\mathbf{E} = \mathbf{i}E_x + \mathbf{j}E_y$. Here $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors along the axes x, y, z, respectively.

The kinetic equation (9.1) yields the equation for the mean velocity of the directional motion of an electron; $\mathbf{w} = N_e^{-1}\int \mathbf{v} f\, d\mathbf{v}$. We multiply Eq. (9.1) by the electron velocity, divide it by the electron density, and integrate over the electron velocities. The first term gives

$$\frac{1}{N_e}\int \mathbf{v}\frac{\partial f}{\partial t}\,d\mathbf{v} = \frac{d}{dt}\left(\frac{1}{N_e}\int \mathbf{v} f\,d\mathbf{v}\right) = \frac{d\mathbf{w}}{dt}$$

Using expression (9.2) for the force in the second term in the modified Eq. (9.1) and integrating by parts, we get for this term the following formula:

$$\frac{e}{mN_e}\exp(i\omega t)\int \mathbf{v}\left(\mathbf{E}\frac{\partial f}{\partial \mathbf{v}}\right)d\mathbf{v} = -\frac{e\mathbf{E}}{mN_e}\exp(i\omega t)\int f\,d\mathbf{v}$$
$$= -\frac{e\mathbf{E}}{m}\exp(i\omega t)$$

The third term transforms into

$$\frac{e}{mcN_e}\int \mathbf{v}\left((\mathbf{v}\times\mathbf{H})\frac{\partial f}{\partial \mathbf{v}}\right)d\mathbf{v}$$
$$= \frac{eH}{mcN_e}\int \mathbf{v}\left(v_y\frac{\partial f}{\partial v_x} - v_x\frac{\partial f}{\partial v_y}\right)d\mathbf{v} = -\frac{e}{mc}\,\mathbf{w}\times\mathbf{H}$$

The right-hand side of the equation can be transformed as follows, taking into account that f_0 is a spherically symmetrical function and τ is independent of the velocity:

$$-\frac{1}{N_e}\int\frac{(f-f_0)}{\tau}\mathbf{v}\,d\mathbf{v}=-\frac{\mathbf{w}}{\tau}$$

Finally, we obtain the following equation for the mean velocity of the electron:

$$\frac{d\mathbf{w}}{dt}+\frac{\mathbf{w}}{\tau}=\frac{\mathbf{F}}{m} \qquad (9.3)$$

Equation (9.3) is the equation of motion for the electron travelling in the field of an external force **F**. The collisions between the electron and the gas particles are accounted for by the frictional force $m\mathbf{w}/\tau$.

The conductivity of a weakly ionized gas. Let us write the components of Eq. (9.3). We denote by $\omega_0 = eH/(mc)$ the so-called cyclotron frequency and introduce the parameters $a_x = eE_x/m$ and $a_z = eE_z/m$. Hence, we obtain the following system of equations:

$$\frac{dw_x}{dt}+\frac{w_x}{\tau}=a_x\exp(i\omega t)+\omega_0 w_y,$$

$$\frac{dw_y}{dt}+\frac{w_y}{\tau}=-\omega_0 w_x,$$

$$\frac{dw_z}{dt}+\frac{w_z}{\tau}=a_z\exp(i\omega t)$$

We wish to find the steady-state solution of this system of equations, which does not depend on the initial conditions. It has the following form:

$$\left.\begin{aligned} w_x&=\frac{i\omega+\tau^{-1}}{\tau^{-2}+(\omega_0^2-\omega^2)+2i\omega\tau^{-1}}a_x\exp(i\omega t)\\ w_y&=\frac{\omega_0}{\tau^{-2}+(\omega_0^2-\omega^2)+2i\omega\tau^{-1}}a_x\exp(i\omega t)\\ w_z&=\frac{1}{i\omega+\tau^{-1}}a_z\exp(i\omega t)\end{aligned}\right\} \qquad (9.4)$$

The solution (9.4) may be used for analyzing the behaviour of a weakly ionized plasma in the external electric and magnetic fields. The electric properties of a weakly ionized plasma are due to the motion of electrons whose mean velocity is given by Eq. (9.4). For instance, for a constant electric field ($\omega = \omega_0 = 0$), Eqs. (9.4) yield

$$\mathbf{w} = \frac{e\mathbf{E}}{m}\tau \qquad (9.5)$$

and the electron mobility is

$$K_e = \frac{e}{m}\tau \qquad (9.6)$$

which is in agreement with Eq. (8.4): $\tau^{-1} \sim N\sigma\,(T/m)^{1/2}$. The conductivity of a weakly ionized plasma in this case is given by Eq. (8.5):

$$\Sigma_0 = N_e e^2 \tau / m \qquad (9.7)$$

which agrees with Eq. (8.6).

The Hall effect. When a magnetic field is applied to a weakly ionized plasma, its conductivity ceases to be a scalar. The motion of electrons under the action of the magnetic field gives rise to electric current perpendicular to the electric field, apart from the current parallel to the electric field. The electric current in the weakly ionized plasma in the direction **i** is $\sum\limits_k E_k \Sigma_{ik}$ where E_k is the electric field strength in the direction of **k**, and Σ_{ik} is the conductivity tensor. According to Eqs. (9.4), this tensor has the form

$$\Sigma_0 \left| \begin{array}{ccc} \dfrac{1+i\omega\tau}{1+(\omega_0^2-\omega^2)\,\tau^2+2i\omega\tau} & \dfrac{\omega_0\tau}{1+(\omega_0^2-\omega^2)\,\tau^2+2i\omega\tau} & 0 \\ \dfrac{\omega_0\tau}{1+(\omega_0^2-\omega^2)\,\tau^2+2i\omega\tau} & \dfrac{1+i\omega\tau}{1+(\omega_0^2-\omega^2)\,\tau^2+2i\omega\tau} & 0 \\ 0 & 0 & \dfrac{1}{1+i\omega\tau} \end{array} \right| \qquad (9.8)$$

where Σ_0 is the conductivity in a constant electric field [see Eq. (9.7)]. The imaginary terms in the conductivity tensor describe the phase shift of the electric current in the plasma with respect to the electric field. We can now use (9.8) to analyze various limiting cases.

(i) *A weakly ionized gas in a variable electric field* ($\omega_0 = 0$). The conductivity in this case is $\Sigma = \Sigma_0/(1 + i\omega\tau)$. If the variation frequency of the electric field is small ($\omega\tau \ll 1$), an electron is slowed down because of the collisions with the gas particles, and the conductivity of the plasma is determined by the conductivity Σ_0 for a constant electric field. For high electric field frequencies the electron changes its direction of motion because of the variation of the polarity of the electric field. Since collisions with the gas particles in this case are not significant, τ does not enter the expression for the conductivity:

$$\Sigma = \frac{N_e e^2}{im\omega} \tag{9.9}$$

Because the conductivity is an imaginary quantity, the phase lag between the electric current and the field is $\pi/2$.

(ii) *A weakly ionized gas in skewed constant electric and magnetic fields.* In this case the current has two components: one along the electric field and the other perpendicular to the electric and magnetic fields; the respective components of the conductivity tensor are

$$\Sigma_{\parallel} = \Sigma_0 \frac{1}{1+\omega_0^2\tau^2}, \quad \Sigma_{\perp} = \Sigma_0 \frac{\omega_0\tau}{1+\omega_0^2\tau^2} \tag{9.10}$$

For high magnetic fields ($\omega_0\tau \gg 1$)* the resulting current is perpendicular to the electric and magnetic fields. The conductivity in this limiting case does not depend on the collision time τ and is given by

$$\Sigma_{\perp} = \frac{\Sigma_0}{\omega_0\tau} = \frac{N_e e^2}{m\omega_0} \tag{9.11}$$

If the transverse electric current does not pass the boundary of plasma, it results in a separation of charges, thus giving rise to an electric field that stops the electrons. This gives rise to a potential difference in the direction perpendicular to the electric and magnetic fields. The phenomenon is known as the Hall effect.

The cyclotron resonance. When $\omega\tau \gg 1$ and $\omega_0\tau \gg 1$, the conductivities in the directions parallel or perpendicular

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* We assume that $\omega_0\tau \ll M/m$ where M is the ion mass, and m is the electron mass. Otherwise, the ions will drift in the same direction and with the same velocity as the electrons do, and the conductivity of plasma will be small compared to the value given by Eq. (9.11).

to the electric field are

$$\left.\begin{aligned}\Sigma_{\parallel} &= \frac{\Sigma_0}{2\left[1-\dfrac{i(\omega_0^2-\omega^2)}{2\omega_0\tau^{-1}}\right]},\\ \Sigma_{\perp} &= \frac{\Sigma_0}{2\left[\dfrac{\omega}{\omega_0}-\dfrac{i(\omega_0^2-\omega^2)}{2\omega_0\tau^{-1}}\right]}\end{aligned}\right\} \quad (9.12)$$

The conductivity is seen to have a resonance at $\omega = \omega_0$, and its value is $\Sigma_{\parallel} = \Sigma_{\perp} = \Sigma_0/2$. The resonance width is $\Delta\omega \sim 1/\tau$. This conductivity resonance at an electric field frequency ω equal to the cyclotron frequency ω_0 is called the cyclotron resonance.

The cyclotron resonance has a simple physical meaning. In a magnetic field, an electron travels along a circular orbit with the cyclotron frequency ω_0. If an electric field is applied in the plane of circular orbit and if this field varies so that its direction remains parallel to the electron velocity, the electron continuously receives energy from the field. Similar to electron motion in the constant electric field, the electron is accelerated until it collides with an atom. Hence, in both cases the conductivities are of the same order of magnitude and are expressed in terms of the frequency $1/\tau$ of collisions between the electron and the atom. If $\omega \neq \omega_0$, the conductivity is considerably lower since the conditions of interaction between the electron and the field are not optimal.

The mean electron energy. Let us estimate the mean energy of electrons travelling in a gas in a constant electric field. Since the electrons take on energy from the electric field, their mean energy can be much higher than the thermal energy of the gas particles. When determining the mean energy of electrons, we shall take into account only their elastic collisions with the gas particles.

To do this we have to write the balance equation for the electron energy since the electrons obtain energy from the electric field and dissipate it in the collisions with the gas particles. Let us determine the mean energy given up by the electron and obtained by the gas particle in the elastic collision between them. This energy is

$$\Delta\varepsilon = \frac{(\mathbf{P}+\Delta\mathbf{P})^2}{2M} - \frac{P^2}{2M} = \frac{(\mathbf{P}\Delta\mathbf{P})}{M} + \frac{(\Delta\mathbf{P})^2}{2M}$$

$$= -\frac{(\mathbf{P}\Delta\mathbf{p})}{M} + \frac{(\Delta\mathbf{p})^2}{2M}$$

Here $\mathbf{P}$ is the momentum of the gas particle before the collision, $\Delta\mathbf{P}$ is the variation of the momentum of the gas particle due to the collision, $\Delta\mathbf{p}$ is the variation of the momentum of the electron due to the collision (conservation of momentum yields $\Delta\mathbf{P} = -\Delta\mathbf{p}$), and M is the mass of the gas particle.

The mass of the gas particle is considerably larger than the electron mass, so that in calculating the electron momentum we can assume the gas particle to be at rest and to have an infinite mass. Therefore, the momentum $\mathbf{p}$ of the electron and its variation $\Delta\mathbf{p}$ in collision with the gas particle do not depend on the momentum $\mathbf{P}$ of the gas particle. Averaging of the expression for $\Delta\varepsilon$ over all the directions of the electron momentum yields $\Delta\varepsilon = \Delta p^2/(2M)$. Since $\Delta p \sim p$ and $p \sim (m\bar{\varepsilon})^{1/2}$ where p is the electron momentum and $\bar{\varepsilon}$ is the mean electron energy, the energy given by the electron to the gas particle in one collision is $\Delta\varepsilon \sim m\bar{\varepsilon}/M$, and the energy given up by the electron per unit time is $\Delta\varepsilon/\tau \sim m\bar{\varepsilon}/(M\tau)$.

The energy obtained by the electron per unit time from the electric field is $e(\mathbf{E}\mathbf{w})$ where $\mathbf{w}$ is the velocity of the directional motion of the electron given by Eq. (9.5): $\mathbf{w} = e\mathbf{E}\tau/m$. Equating the energies obtained and dissipated by the electron per unit time, we find the following estimate for the mean electron energy:

$$\bar{\varepsilon} \sim M\left(\frac{eE\tau}{m}\right)^2 \qquad (9.13)$$

This estimate holds if the mean electron energy is high compared to the thermal energy of the gas particles, so that the electron only gives up the energy to the gas, and also if the mean energy is small compared to the characteristic electron energies at which there is a significant contribution from inelastic processes.

Using a similar approach, let us find the mean energy of tho electron travelling in the gas in scewed constant electric and magnetic fields. Equation (9.4) indicates that the electron takes on from the field the energy $eEw=\tau(eE)^2\times\times(1+\omega_0^2\tau^2)^{-1}/m$ so the order-of-magnitude estimate of the mean electron energy is

$$\bar{\varepsilon}\sim M\left(\frac{eE}{m}\right)^2\left(\frac{1}{\tau^2}+\omega_0^2\right)^{-1} \tag{9.14}$$

Let us, finally, consider the motion of an electron in the gas in the variable electric field. If the electric field is $E\cos\omega t$, we obtain the following expression for the drift velocity of the electron along the field by taking the real part of Eq. (9.4):

$$w_x=\frac{eE}{m}\left(\omega^2+\frac{1}{\tau^2}\right)^{-1}\left(\frac{1}{\tau}\cos\omega t+\omega\sin\omega t\right)$$

Hence, we find the energy obtained by the electron per unit time (averaged over time):

$$\overline{eEw_x}=\frac{(eE)^2}{2m\tau}\left(\omega^2+\frac{1}{\tau^2}\right)^{-1}$$

Thus, we find the mean energy of an electron travelling in a gas in a variable electric field:

$$\bar{\varepsilon}\sim M\left(\frac{eE}{m}\right)^2\left(\frac{1}{\tau^2}+\omega^2\right)^{-1} \tag{9.15}$$

The magnetohydrodynamic equations. If the plasma density is sufficiently high, we must take into account the fields produced by plasma motion. These fields, which are due to distribution and motion of plasma, affect the motion of plasma, that is, the plasma parameters and the fields produced by plasma are interrelated. The motion of plasma and variation of its parameters can be described by the continuity equation for the density of electrons and ions (6.6), the equation for the mean momentum of the electrons and ions (6.8), Poisson's equation (3.2), and the Maxwell equations. The resulting system is called the system of equations of magnetohydrodynamics:

$$\left.\begin{aligned}\frac{\partial N}{\partial t}+\operatorname{div}(N\mathbf{w})=0,\\ \frac{\partial\mathbf{w}}{\partial t}+(\mathbf{w}\nabla)\mathbf{w}+\frac{\operatorname{grad}p}{MN}-\frac{\mathbf{F}}{M}=0,\\ \operatorname{div}\mathbf{E}=4\pi e(N_i-N_e),\\ \operatorname{curl}\mathbf{H}=\frac{4\pi}{c}\mathbf{j}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t},\\ \operatorname{curl}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t},\\ \operatorname{div}\mathbf{H}=0\end{aligned}\right\} \qquad (9.16)$$

The first two equations can be written both for the electrons and for the ions; therefore, **w** and N are the velocity and the density of the electrons or the ions, respectively. To (9.16) we must add the equation of state of the type (6.14) and the thermodynamic equation for the process (for instance, the adiabatic equation of the process if the variation of the parameters of the system is an adiabatic process). These equations interrelate the density, temperature, and pressure of the plasma particles. We must add to these equations also Ohm's law relating the plasma current and the electric field strength. The system of magnetohydrodynamic equations with the addition of the above-mentioned equations and the initial conditions will give a definite description of plasma development.

Let us analyze the plasma motion in which the velocity of electrons is considerably greater than the velocity of ions. Then the electric current is due to the electrons and is given by

$$\mathbf{j}=-eN_e\mathbf{w}$$

where **w** is the directional velocity of the electrons, and N_e is their density. If the motion occurs in a magnetic field, an additional electric field is produced in the laboratory frame of reference, the field given by

$$\mathbf{E}'=\frac{1}{c}\mathbf{w}\times\mathbf{H}=-\frac{1}{ecN_e}\mathbf{j}\times\mathbf{H} \qquad (9.17)$$

This field acts on the electrons giving rise to an additional force acting on the entire plasma. The force acting on the unit volume of plasma is

$$e\mathbf{E}'N_e = -\frac{1}{c}\,\mathbf{j}\times\mathbf{H} \qquad (9.18)$$

If the plasma conductivity is sufficiently high, its response to the electric field (9.17) will result in the movement of electrons. This movement will continue until separation of the electrons and ions gives rise to an internal electric field in plasma

$$\mathbf{E} = -\frac{1}{c}\,\mathbf{w}\times\mathbf{H} \qquad (9.19)$$

which will compensate the field (9.17). We insert Eq. (9.19) into the Maxwell equation $-(\partial\mathbf{H}/\partial t) = c$ curl $\mathbf{E}$ and obtain the following equation for the variation of the magnetic field:

$$\frac{\partial\mathbf{H}}{\partial t} = \operatorname{curl}(\mathbf{w}\times\mathbf{H}) \qquad (9.20)$$

Let us analyze the variation of the magnetic field and the plasma motion when the electric current is due to the electrons and the plasma conductivity is high. Transform Eq. (9.20) by writing curl $(\mathbf{w}\times\mathbf{H}) = \mathbf{w}\operatorname{div}\mathbf{H} + (\mathbf{H}\nabla)\mathbf{w} - (\mathbf{w}\nabla)\mathbf{H} - \mathbf{H}\operatorname{div}\mathbf{w}$, noting that according to the Maxwell equation div $\mathbf{H} = 0$, and using the expression for div $\mathbf{w}$ from the continuity equation for the electrons, $N_e \times \operatorname{div}\mathbf{w} = -(\partial N_e/\partial t) - (\mathbf{w}\nabla)N_e$. We obtain

$$\frac{\partial\mathbf{H}}{\partial t} - \frac{\mathbf{H}}{N_e}\frac{\partial N_e}{\partial t} + (\mathbf{w}\nabla)\mathbf{H} - \frac{\mathbf{H}}{N_e}(\mathbf{w}\nabla)N_e = (\mathbf{H}\nabla)\mathbf{w}$$

We divide this equation by N_e and find that

$$\frac{d}{dt}\left(\frac{\mathbf{H}}{N_e}\right) = \left(\frac{\mathbf{H}}{N_e}\nabla\right)\mathbf{w} \qquad (9.21)$$

where

$$\frac{d}{dt}\left(\frac{\mathbf{H}}{N_e}\right) = \frac{\partial}{\partial t}\left(\frac{\mathbf{H}}{N_e}\right) + (\mathbf{w}\nabla)\frac{\mathbf{H}}{N_e}$$

is the derivative at the point which travels together with the plasma.

Let us analyze the motion of the elementary plasma volume with the length $d\mathbf{l}$ and the cross section ds which contains $N_e\,ds\,d\mathbf{l}$ electrons. At first, assume that the vector $d\mathbf{l}$ is parallel to the magnetic field $\mathbf{H}$ so that the magnetic flux through this elementary plasma volume is $H ds$. Let us determine the time dependence of the magnetic flux. If the plasma velocity at one end of the segment $d\mathbf{l}$ is $\mathbf{w}$, then at the other end the velocity is $\mathbf{w} + (d\mathbf{l}\,\nabla)\mathbf{w}$ so that the variation of the segment length during a small time interval δt is $\delta t\,(d\mathbf{l}\,\nabla)\mathbf{w}$. Hence, the length of the segment satisfies the equation

$$\frac{d}{dt}(d\mathbf{l}) = (d\mathbf{l}\nabla)\,\mathbf{w}$$

which is identical to Eq. (9.21). This fact indicates that (1) at all times the segment $d\mathbf{l}$ has the same direction as the magnetic field and (2) the length of the plasma element remains proportional to the quantity H/N_e, that is, the magnetic flux through this plasma element does not vary with time during plasma motion. Thus, the lines of magnetic force are "frozen" into the plasma, that is, their direction is such that the plasma electrons travel along these lines. Remember that this is the case when the plasma conductivity is high.

Let us consider the steady-state motion of a high-conductivity plasma. According to Eq. (9.19), each plasma electron is under the action of a force $\mathbf{F} = -e\mathbf{E} = (e/c)\,\mathbf{w}\times\mathbf{H} = -(cN_e)^{-1}\,\mathbf{j}\times\mathbf{H}$. Inserting into the expression for the force the current density $\mathbf{j} = c\,(4\pi)^{-1}$ curl $\mathbf{H}$, we obtain

$$\begin{aligned}\mathbf{F} &= -\frac{1}{cN_e}\,\mathbf{j}\times\mathbf{H} = \frac{1}{4\pi N_e}\,\mathbf{H}\times\operatorname{curl}\mathbf{H}\\ &= \frac{1}{4\pi N_e}\left[\frac{1}{2}\operatorname{grad}H^2 - (\mathbf{H}\nabla)\,\mathbf{H}\right]\end{aligned} \qquad (9.22)$$

Let us substitute Eq. (9.22) in the second equation of the system (9.16). We assume that the velocity of the directional motion of the electrons is considerably greater than their thermal velocity. Hence, we can neglect the term $(\mathbf{w}\nabla)\mathbf{w}$ compared to the term $(MN_e)^{-1}\operatorname{grad}p$ and obtain the following equation:

$$\operatorname{grad}\left(p + \frac{H^2}{8\pi}\right) - \frac{1}{4\pi}(\mathbf{H}\nabla)\,\mathbf{H} = 0 \qquad (9.23)$$

The quantity $H^2/(8\pi)$ is termed the magnetic field pressure; it is the pressure with which the magnetic field acts on the plasma.

Let us analyze the properties of a cylindrical plasma column maintained by a direct current. Here the lines of magnetic force are straight lines and Eq. (9.23) in the direction perpendicular to the field has the following form: $\mathrm{grad}_\perp\, [p + H^2/(8\pi)] = 0$. This equation shows that the total pressure $p + H^2/(8\pi)$, which is the sum of the gas-kinetic pressure and the magnetic field pressure, is independent of the transverse coordinate. Let the radius of the plasma column be a and the current in it be I so that the magnetic field at the surface of the column is $H = 2I/(ca)$. The total pressure outside the column near its surface equals the magnetic field pressure $I^2/(2\pi c^2 a^2)$, and the total pressure inside the plasma column equals the gas-kinetic pressure p. Equating these two pressures, we find the radius of the plasma column

$$a = \frac{I}{c\,(2\pi p)^{1/2}} \qquad (9.24)$$

An increase in the current in the plasma column accompanied by the respective increase in the magnetic field is seen to give rise to a contraction of the plasma column. This phenomenon is called the pinch effect, and the state of the plasma column itself is known as the z pinch.

10 Waves in a Plasma

Although plasma, as a consequence of the interaction between particles, is a gas, oscillations and noises play a much greater role than in ordinary gas. First, in a plasma that is located in external fields and is not homogeneous, a wide variety of oscillations can occur because of the long-range interaction between particles. Secondly, these oscillations vary frequently and become amplified to a relatively high energy. In this case, the plasma oscillations determine its parameters and development. We shall analyze the simplest types of oscillations in gas and plasma.

Acoustic oscillations. The natural vibrations of gas are the acoustic vibrations, that is, the waves of compression and rarefaction which propagate in gas. We have to find the relationship between the frequency ω of the vibrations and their wavelength λ, which is related to the wave vector $\mathbf{k}$ as follows: $k = 2\pi/\lambda$.

In our analysis, we shall assume that the amplitude of the system's oscillations is small. Thus, any macroscopic parameter of the system can be expressed as

$$A = A_0 + \sum_{\omega} A'_{\omega} \exp[i(kx - \omega t)] \qquad (10.1)$$

where A_0 is the macroscopic parameter in the absence of vibrations, A'_{ω} is the amplitude of vibrations, ω is the frequency of vibrations, and k is the respective wave number (the wave propagates along the x axis). Since the vibration amplitude is small, the vibration with a given amplitude does not depend on the amplitudes for other frequencies. That is to say, there is no interaction between waves of different frequencies owing to the smallness of their amplitudes. Therefore, we can retain only one term in the sum in Eq.(10.1) and express the macroscopic parameter A in the following form:

$$A = A_0 + A' \exp[i(kx - \omega t)] \qquad (10.2)$$

In our analysis of the acoustic oscillations in gas let us write the gas density N, the gas pressure p, and the mean gas velocity w in the form (10.2); we shall assume further that the gas is at rest, that is, $w_0 = 0$. Using the continuity equation (6.6) and neglecting the terms with squared oscillation amplitudes, we obtain

$$\omega N' = kN_0 w' \qquad (10.3)$$

Note that the velocity $\mathbf{w}'$ of gas propagation with the acoustic wave is directed along the wave vector $\mathbf{k}$.

Similarly, the Euler equation (6.15) yields, up to quadratic terms, the following equation:

$$\omega w' = \frac{kp'}{MN_0} \qquad (10.4)$$

Assume that the regions of compression and rarefaction of gas do not have time to exchange energy during propagation of the acoustic wave, that is, the process is adiabatic. The larger the wavelength, the better satisfied are the adiabaticity conditions. The parameters of the acoustic wave satisfy the adiabatic equation

$$pN^{-\gamma} = \text{const} \qquad (10.5)$$

where $\gamma = c_p/c_V$ is the adiabatic exponent, c_p is the specific heat at constant pressure, and c_V is the specific heat at constant volume. Using the expansion (10.2), we find the following relationship between the wave parameters:

$$\frac{p'}{p_0} = \gamma \frac{N'}{N_0} \qquad (10.6)$$

Eliminate the amplitudes from Eqs. (10.3), (10.4), and (10.6) and make use of the equation of state (6.14), $p_0 = NT$. The result is the relationship between the acoustic frequency ω and the wave number k:

$$\omega = \left(\gamma \frac{T}{M}\right)^{1/2} k \qquad (10.7)$$

Equation of the type (10.7), which relates the wave frequency to the wave number, is called the dispersion relation. The sound propagation velocity $(\partial\omega/\partial k) = (\gamma T/M)^{1/2}$, which is called the group velocity, is seen to be of the order of the thermal velocity of the gas particles.

Plasma oscillations. Let us analyze the oscillations which are due to the motion of charged particles in a plasma or a weakly ionized gas. In the simplest case of a homogeneous plasma and in the absence of external fields, there are two kinds of the natural plasma oscillations since plasma has two species of charged particles. These kinds of oscillations differ considerably since the electrons and ions responsible for them differ greatly in mass.

Let us study the high-frequency oscillations of the homogeneous plasma. These oscillations are due to electron motion; they are referred to as plasma waves. Because of their large mass, the ions are not involved in these oscillations, and when analyzing plasma waves, we shall assume the ions to be at rest and their charge uniformly distributed over the gas volume.

Similar to acoustic oscillations in gas, we shall derive the dispersion relation for the plasma waves from the continuity equation (6.6), the Euler equation (6.15), and the adiabatic equation (10.5) for the wave. Moreover, we must take into account the electric field produced by the motion of electrons owing to disturbance of the quasineutrality of plasma. We shall introduce the electric field term into the Euler equation (6.15) while the electric field strength will be given by Poisson's equation (3.2).

Similar to the derivation of the dispersion relation for acoustic oscillations, let us assume further that the macroscopic parameters of the oscillating plasma can be written in the form (10.2) and that in the absence of oscillations the mean velocity w of electrons and the electric field strength E are zero. Hence, we obtain

$$\left.\begin{aligned} -i\omega N_e' + ikN_0 w' &= 0, \\ -i\omega w' + i\frac{kp'}{mN_0} + \frac{eE'}{m} &= 0, \\ \frac{p'}{p_0} &= \gamma\frac{N_e'}{N_0}, \\ ikE' &= -4\pi eN_e' \end{aligned}\right\} \qquad (10.8)$$

Here k and ω are the wave number and the frequency of the plasma oscillations; N_0 is the mean density of charged particles, $p_0 = N_0 m\langle v_x^2\rangle$ is the electron gas pressure in the absence of oscillations, m is the electron mass, v_x is the electron velocity component in the direction of oscillations, and the angular brackets denote averaging over the electron velocities. The quantities N_e', w', p', and E' in Eq. (10.8) are the oscillation amplitudes of the electron density, mean velocity, pressure, and electric field strength, respectively.

Eliminate the oscillation amplitudes of the macroscopic quantities from the system of equations (10.8). Hence, we obtain the following dispersion relation for the plasma oscillations:

$$\omega^2 = \omega_0^2 + \gamma\langle v_x^2\rangle k^2 \qquad (10.9)$$

where $\omega_0 = (4\pi N_0 e^2/m)^{1/2}$ is the plasma frequency [see Eq. (3.9)].

Note that plasma oscillations are longitudinal in contrast to electromagnetic oscillations. Hence, the electric field due to plasma waves is directed along the wave vector. This fact was used in deriving the system of equations (10.8).

Ion sound. We consider now the oscillations that are due to the motion of the ions in a uniform isotropic plasma. The peculiarity of these oscillations is determined by the large mass of the ions. Owing to its small mass, an electron can follow the motion of an ion so that the plasma remains quasineutral in any large volume containing a large number of charged particles:

$$N_e = N_i$$

Moreover, the electrons have time to redistribute due to the electric field in the plasma; this distribution is given by the Boltzmann formula

$$N_e = N_0 \exp (e\varphi/T) \approx N_0 (1 + e\varphi/T)$$

where φ is the electric potential due to the oscillation of plasma. Hence, we obtain the amplitude of oscillation of the ion density:

$$N_i' = N_0 \frac{e\varphi}{T_e} \qquad (10.10a)$$

where T_e is the electron temperature.

Let us use the motion equations for ions. The continuity equation

$$\frac{\partial N_i}{\partial t} + \frac{\partial}{\partial x} (N_i w_i) = 0$$

yields

$$\omega N_i' = k N_0 w_i \qquad (10.10b)$$

where ω is the frequency, k is the wave number, and w_i is the mean ion velocity due to the oscillations. Here we assume the usual harmonic dependence of the oscillation parameters on position and time [see Eq. (10.2)].

Furthermore, let us make use of the equation of motion for ions due to the electric field of the wave $M (d\mathbf{w}_i/dt) = e\mathbf{E} = - e \operatorname{grad} \varphi$ where M is the ion mass. Taking into account the harmonic dependence (10.2) on position and time, we obtain

$$M \omega w_i = e k \varphi \qquad (10.10c)$$

When we eliminate the oscillation amplitudes of N_i', φ, and w_i in the system of equations (10.10), we obtain the following dispersion relation between the frequency and the wave number:

$$\omega = \left(\frac{T_e}{M}\right)^{1/2} k \qquad (10.11)$$

The oscillations caused by the ion motion are known as the ion sound. Similar to the plasma oscillations, the ion sound is a longitudinal wave, that is, the wave vector, **k**, is parallel to the oscillating vector of the electric field, **E**. The dispersion relation for the ion sound is similar to that for ordinary sound. This is due to the fact that both types of oscillations are characterized by a short-range interaction. In the case of the ion sound, the interaction is short-ranged because the electric field of the propagating wave is screened by the plasma. This screening is effective if the wavelength of the ion sound is considerably larger than the Debye-Hückel radius for the plasma where the sound propagation occurs:

$$kr_D \ll 1$$

The dispersion relation (10.11) for the ion sound is valid if this condition is satisfied.

Magnetohydrodynamic waves. Let us study the waves developing in a high-conductivity plasma which travels in a magnetic field. The magnetic lines of force in this case are "frozen" into the plasma so that if the plasma current is "shifted" somewhat so as to displace the magnetic lines of force, the magnetic field acts so as to shift the plasma current back. This gives rise to waves known as magnetohydrodynamic waves.

Consider the magnetohydrodynamic waves whose wavelength is much less than the radius of curvature of the magnetic field:

$$\frac{1}{k} \ll \frac{H}{|\operatorname{grad} H|}$$

where k is the wave number, and H is the magnetic field strength. Hence, the magnetic force lines can be assumed to be straight. The system of magnetohydrodynamic equations (9.16) can be used to describe the waves. As above, we write the plasma parameters as sums of two terms:

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{h} \exp [i (\mathbf{kr} - \omega t)],$$
$$N = N_0 + N' \exp [i (\mathbf{kr} - \omega t)],$$
$$p = p_0 + p' \exp [i (\mathbf{kr} - \omega t)]$$

and also

$$w = w' \exp [i (\mathbf{kr} - \omega t)]$$

The first term refers to the plasma in the absence of oscillations and also when the oscillation amplitude is small and does not depend on position and time. Furthermore, we assume that the plasma is at rest in the absence of oscillations and its pressure is zero, that is, small compared to the magnetic field pressure. Inserting the plasma parameters in the above form into the system of magnetohydrodynamic equations (9.16), we obtain the following system of equations for the oscillation amplitudes of the plasma parameters:

$$\left.\begin{aligned} \mathbf{kh} = 0, \quad -\omega \mathbf{h} &= \mathbf{k} \times (\mathbf{w} \times \mathbf{H}_0), \\ \omega N' &= \mathbf{kw} N_0, \\ \omega \mathbf{w} &= \frac{1}{4\pi N_0} \mathbf{H}_0 \times (\mathbf{k} \times \mathbf{h}) \end{aligned}\right\} \qquad (10.12)$$

Introduce a new coordinate system in which the wave vector **k** is parallel to the x axis and the magnetic field strength $\mathbf{H}_0$ is in the xy plane. The first equation (10.12) indicates that magnetic field amplitude **h** has two components: h_y and h_z. Then we write the remaining equations (10.12) in components:

$$\left.\begin{aligned} -\omega h_z &= k w_z H_{0x}, \\ -\omega h_y &= k (w_y H_{0x} - w_x H_{0y}), \\ \omega N' &= k w_x N_0, \\ \omega w_z &= -k \frac{H_{0x}}{4\pi N_0} h_z, \\ \omega w_y &= -k \frac{H_{0x}}{4\pi N_0} h_y, \\ \omega w_x &= k \frac{H_{0y}}{4\pi N_0} h_y \end{aligned}\right\} \qquad (10.13)$$

The system (10.13) can be divided into two independent systems. One of these systems corresponds to magnetic field oscillations along the z axis. The oscillation frequency is related to the wave number by the following dispersion relation:

$$\omega = \frac{H_0 k}{(4\pi N_0)^{1/2}} \qquad (10.14)$$

Another system of equations describes the oscillations in the xy plane; it yields the same dispersion relation:

$$\omega = \frac{H_0 k}{(4\pi N_0)^{1/2}} \qquad (10.15)$$

Thus, when the wave travels perpendicular to the magnetic field, the dispersion relations for the waves of both types are the same. The phase velocity of the wave coincides with the group velocity and both are equal to

$$u = \frac{H_0}{(4\pi N_0)^{1/2}} \qquad (10.16)$$

The velocity u is known as the Alfven velocity, and the magnetohydrodynamic waves are known as the Alfven waves.

Propagation of electromagnetic waves in a plasma. Let us derive the dispersion relation for the electromagnetic wave propagating in a plasma. The plasma exerts the following effect on the propagation of the electromagnetic wave. The electric field of the wave gives rise to an electron current which produces a magnetic field which, in its turn, affects the field of the wave. Make use of the Maxwell equations for the electromagnetic wave:

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \quad \operatorname{curl} \mathbf{H} = \frac{4\pi}{c}\mathbf{j} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \qquad (10.17)$$

Here $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields in the electromagnetic wave, $\mathbf{j}$ is the density of the electron current produced by the effect of the wave's field, and c is the velocity of light.

Applying the operator $-c^{-1}(\partial/\partial t)$ to the first equation of (10.17) and the operator curl to the second equation and eliminating the magnetic field from the resulting equations, we obtain the following equation:

$$\operatorname{grad}\operatorname{div}\mathbf{E}-\nabla^2\mathbf{E}\frac{4\pi}{c^2}\frac{\partial \mathbf{j}}{\partial t}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0$$

We assume that the plasma is quasineutral and that div $\mathbf{E} = 0$ according to Poisson's equation (3.2). The electric current $\mathbf{j}$ is due to the motion of the electrons: $\mathbf{j} = -eN_0\mathbf{w}$ where N_0 is the density of electrons, and $\mathbf{w}$ is the electron velocity due to the effect of the electromagnetic wave. The equation of motion of the electron is $m(d\mathbf{w}/dt) = -e\mathbf{E}$ so that

$$\frac{\partial \mathbf{j}}{\partial t}=-eN_0\frac{d\mathbf{w}}{dt}=\frac{e^2N_0}{m}\mathbf{E}$$

Hence, we obtain the following equation for the electric field of the electromagnetic wave:

$$\nabla^2\mathbf{E}-\frac{\omega_0^2}{c^2}\mathbf{E}+\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0$$

where ω_0 is the plasma frequency.

Write the electric field in the form (10.2) and substitute it into the above equation; the result is the dispersion relation for the frequency ω and the wave number k of the electromagnetic wave propagating in the plasma:

$$\omega^2=\omega_0^2+c^2k^2 \qquad (10.18)$$

If the plasma density is low ($N_0 \rightarrow 0$, $\omega_0 \rightarrow 0$), Eq. (10.18) is the same as the dispersion relation for the electromagnetic wave propagating in a vacuum: $\omega = kc$. The dispersion relation (10.18) shows that the electromagnetic waves whose frequency is lower than the plasma frequency cannot propagate in a plasma. Such waves are damped in the plasma and the dispersion relation (10.18) shows that the characteristic distance of this damping is of the order $c/(\omega_0^2-\omega^2)^{1/2}$.

Damping of plasma oscillations in a weakly ionized plasma. When deriving the dispersion relation for the plasma oscillations, we neglected the interaction between the electrons and the neutral particles. This interaction results in the fading of the plasma oscillations because the energy of the electron oscillations is transferred to the neutral particles. To account for this interaction in the dispersion relation for the plasma waves, let us use Eq. (6.16), rather than Eq. (6.15), as the equation for the mean electron momentum. Then the second equation in the system (10.8) is transformed into

$$-i\omega w' + \frac{ikp'}{mN_0} + \frac{eE'}{m} = \frac{w'}{\tau} \qquad (10.19)$$

and the remaining equations of this system are not changed. Here τ is the characteristic time of the elastic collision between an electron and a neutral gas particle giving rise to a noticeable change in the electron momentum. We assume that the mean velocity of directional gas motion is zero.

We replace the second equation in (10.8) with Eq. (10.19) and eliminate the amplitudes from the resulting system. The result is the following dispersion relation instead of Eq. (10.9):

$$\omega = (\omega_0^2 + \gamma \langle v_x^2 \rangle k^2)^{1/2} - \frac{i}{\tau} \qquad (10.20)$$

To derive this relation we assumed that

$$\omega\tau \gg 1 \qquad (10.21)$$

Substituting (10.20) into Eq. (10.2), we find that the wave amplitude decreases with time as exp $(-t/\tau)$ and this decrease is due to the scattering of electrons by the gas particles. The condition of the existence of the plasma waves is that the characteristic time of the wave damping must be considerably less than the oscillation period, namely condition (10.21) must hold. The frequency of collisions between an electron and the gas particles is $1/\tau \sim Nv\sigma$ where N is the gas density, v is the characteristic electron velocity, and σ is the cross section of the collision between the electron and gas particles. Assuming that this cross section is of the order of the gas-kinetic cross section ($\sim 10^{-15}$ cm^2), the characteristic electron energy is of the order of 1 eV, and the frequency ω is of the order of the plasma frequency (3.9), we obtain the following estimate for condition (10.21):

$$N_e^{1/2}/N \gg 10^{-12}\ \mathrm{cm}^{3/2}$$

This indicates that in some gas-discharge plasmas the condition of existence of plasma oscillations will not be satisfied.

The interaction between plasma waves and electrons. The above mechanism of damping of plasma waves is due to the collisions between electrons and gas particles. The energy exchange can occur also in the absence of collisions. Let us analyze the interaction between electrons and plasma oscillations. We introduce the frame of reference in which the wave is at rest. This wave captures the electrons whose velocity is close to the wave's velocity (Fig. 17). There is a continuous energy exchange between the wave and the electron captured in the potential well of the wave.

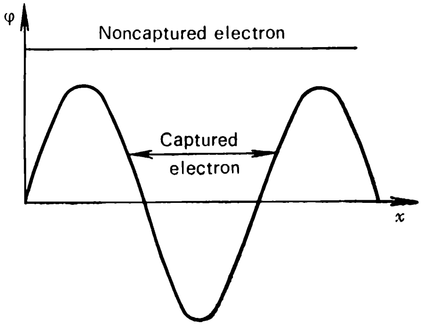


FIG. 17. Interaction of electrons with plasma oscillations.

For instance, if in the frame of reference in which the wave is at rest an electron travels along the wave with a velocity u and reverses its direction of motion in a certain time period, then it transfers to the wave during this time period the following energy:

$$\Delta\varepsilon = \frac{m(v_{\text{ph.}}+u)^2}{2} - \frac{m(v_{\text{ph.}}-u)^2}{2} = 2mv_{\text{ph.}}u$$

Here $v_{\text{ph.}} = \omega/k$ is the phase velocity of the wave. The characteristic velocity of the captured electrons in the frame of reference connected with the wave, u, is of the order of $(e\varphi/m)^{1/2}$ where φ is the amplitude of the wave potential. It may be seen that the energy exchanged between the wave and the noncaptured electron is of the order of $e\varphi$ while for the captured electron this energy is of the order of $v_{\text{ph.}}(me\varphi)^{1/2}$, that is, for small wave amplitudes, considerably higher than the energy exchanged by the noncaptured electron. Hence, when analyzing the energy exchange between electrons and plasma waves, we can consider only the electrons captured by the wave.

Since the captured electron oscillates in the wave's field, the energy exchanged between the electron and the wave must be zero when averaged over a large time interval. But this is not the case when we take into account the collisions between the captured and noncaptured electrons. The energy exchange between these electrons results ultimately in the energy exchange between electrons and the wave. In other words, the interaction between the electrons and the wave alters the distribution function for the electrons with velocities close to the phase velocity $v_{\text{ph.}}$ of the wave. Indeed, the result of the interaction is that the number of electrons with the velocities $v_{\text{ph.}}+u$ and $v_{\text{ph.}}-u$ become equal. However, the collisions tend to restore the distribution function of electrons, and this process results in the energy exchange between the electrons and the wave.

Let us assume that the energy exchange between electrons takes less time than the period of oscillation of the captured electron in the potential well of the wave. The frequency of these oscillations is of the order of $k(e\varphi/m)^{1/2} \sim (eE'k/m)^{1/2} \sim (e^2N_e'/m)^{1/2}$ where k is the wave number, and φ, E', N_e' are the amplitudes of the electric potential, of the electric field strength of the wave, and of the electron density, respectively. To derive this frequency, we made use of the last equation of the system (10.8). The frequency of the energy exchange between the electrons equals the frequency of collisions between the electrons, $N_e v\sigma$, of the order of $N_e(e^4/T^2) \times (T/m)^{1/2}$ [see Eq. (4.14)] where N_e is the electron density, and T is the electron temperature. Hence, we find that the interaction between the electrons and the wave does not alter the distribution function of electrons when

$$\frac{N_e'}{N_e} \ll \frac{N_e e^6}{T^3} \qquad (10.22)$$

Provided this condition is satisfied, let us determine the direction of the energy exchange between the wave and the electrons in a plasma. Now the electron distribution function is not altered owing to the interaction with the wave, and we have to compare the number of electrons with velocity $v_{\text{ph.}}+u$ that transfer the energy to the wave and the number of electrons with a velocity $v_{\mathrm{ph.}}-u$ that take the energy from the wave. The number of captured electrons with a given velocity is proportional to the electron distribution function $f(v)$. Hence, the wave gives its energy to electrons and is damped if $f(v_{\mathrm{ph.}}-u)$ is larger than $f\,(v_{\mathrm{ph.}}+u)$. Since $f(v_{\mathrm{ph.}}\pm u)=f(v_{\mathrm{ph.}})\pm u[\partial f(v_{\mathrm{ph.}})/\partial v_x]$, the wave is damped when

$$\left.\frac{\partial f}{\partial v_x}\right|_{v_x=v_{\mathrm{ph.}}}<0 \qquad (10.23)$$

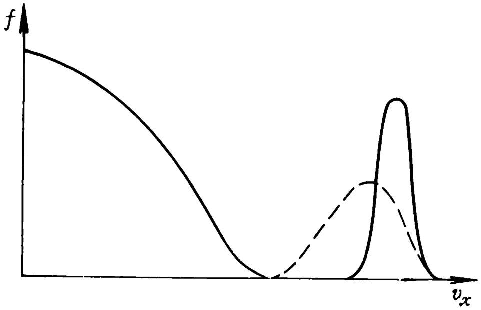
Here v_x is the component of the electron velocity in the direction of the wave propagation, and the derivative is taken for the electron velocity being equal to the phase velocity $v_{\mathrm{ph.}}$ of the wave. When the condition opposite to Eq. (10.23) is satisfied, the wave takes the energy from electrons and its amplitude increases.

In deriving the condition (10.23) of the wave damping, we have used the condition (10.22) which is satisfied when the field of the wave does not affect the distribution function of electrons. However, (10.23) holds also when the condition (10.22) is not satisfied and the electron distribution function is altered by the effect of the wave. The interaction with the wave tends to equalize the number of the electrons with velocities $v_{\mathrm{ph.}}+u$ and $v_{\mathrm{ph.}}-u$ and, hence, it decreases the derivative $\partial f/\partial v_x$ but does not change its sign. Thus, the condition (10.23) is valid even when the interaction with the wave alters the electron distribution function.

The condition (10.23) is satisfied if the electron distribution function has the Maxwellian form or is a monotonically decreasing function, and the oscillations in such a plasma decrease with time owing to the interaction with electrons. However, when an electron beam is injected into the plasma producing the electron distribution function shown in Fig. 18, the plasma oscillations in this system will be amplified getting the energy from the electrons. The interaction between the plasma oscillations and the electrons will reduce the velocity of the electrons until the electron distribution function becomes a monotonically decreasing one.

Amplification of the oscillations means that their amplitude increases with time. While the oscillation amplitude is small and the oscillations do not alter the electron distribution function, that is, while (10.22) is satisfied, the oscillation amplitude increases exponentially. When the plasma particles transfer their energy to the plasma wave, which is thus amplified, such a state of plasma is termed the unstable state.

FIG. 18. The electron distribution function for the electron beam injected into a plasma. The solid line represents the initial distribution and the dashed line is the distribution established after a period of time.



The attenuation factor for waves in plasma. Let us introduce into Eq. (10.2) the attenuation factor γ for the waves so that the oscillation amplitude varies as $\exp(-\gamma t)$. Let us estimate the attenuation factor when the wave in plasma does not affect the distribution of charged particles, that is, a condition of type (10.22) is satisfied. Variation of the energy W of the plasma wave per unit volume per unit time may be estimated as follows:

$$\frac{dW}{dt} \sim \nu \int_{-u_0}^{+u_0} f(v)\,\Delta\varepsilon\,du$$

Here $\nu \sim u_0 k$ is the oscillation frequency for the electron captured in the potential well of the wave, $u_0 = (2e\varphi/m)^{1/2}$, φ is the amplitude of the potential oscillations, and $\Delta\varepsilon = 2mv_{\text{ph.}}u_0$ is the variation of the electron energy when the direction of the electron motion is reversed (we take into account only the interaction between the wave and the captured electrons). The right-hand side of the above relation may be estimated as

$$u_0 k \times \left(\frac{\partial f}{\partial v_x}\right) u_0 \times m v_{\text{ph.}} u_0 \times u_0 \sim \left(\frac{\partial f}{\partial v_x}\right) \frac{e^2\omega}{mk^2} W$$

where we have used the relations $v_{\text{ph.}} = \omega/k$ and $W \sim E'^2 \sim \varphi^2/k^2$ (E' is the amplitude of the wave's electric field). From the definition of the attenuation factor for the plasma wave, $dW/dt = -\gamma W$, we obtain the following estimate:

$$\gamma \sim \frac{e^2\omega}{mk^2}\left|\frac{\partial f}{\partial v_x}\right|. \qquad (10.24)$$

Attenuation occurs when (10.23) is satisfied. The attenuation of type (10.24), which is due to the interaction between charged particles and the wave, is known as the Landau damping.

The condition of existence of the plasma waves and the ion sound has the following form:

$$\gamma \ll \omega \qquad (10.25)$$

Let us transform the condition (10.25) assuming the Maxwell distribution function for the particles. For the plasma oscillations taking $\omega \sim \omega_0$, we obtain

$$kr_D \ll 1 \qquad (10.26)$$

where r_D is the Debye-Hückel radius (3.6) When this condition is satisfied, the phase velocity of the wave is considerably higher than the thermal velocity, so that the electrons captured by the wave are at the tail of the distribution function.

When the ion sound propagates in a plasma in which the temperatures of electrons and ions are the same, the phase velocity of the sound is of the order of the thermal velocity of ions and the attenuation factor is of the order of the sound frequency. Therefore, the ion sound can propagate only in the plasmas in which the electron temperature T_e is considerably higher than the ion temperature T_i:

$$T_e \gg T_i \qquad (10.27)$$

The beam-plasma instability. Assume that an electron beam penetrates a plasma and that the velocity of the electrons in the beam is considerably higher than the thermal velocity of the plasma electrons while the density N_b of the electrons in the beam is considerably lower than the density N_0 of the plasma electrons. Deceleration of the electron beam can occur owing to the scattering of the electrons of the beam on the electrons and ions of the plasma. There is, however, another mechanism of deceleration of the electron beam, which is known as the Langmuir paradox* or beam instability. This kind of deceleration can be more effective than deceleration due to scattering on the charged plasma particles.

This mechanism acts as follows. The plasma oscillations are generated in plasma. Interacting with the electrons of the beam and taking energy from them, these oscillations are amplified. Thus, the energy of the electron beam is transformed into the energy of plasma oscillations and it remains in the plasma. Further, this energy may transfer to other degrees of freedom in the plasma.

Let us analyze the amplification of the plasma oscillations in the above case assuming that the amplitude of oscillations is small and that the temperatures of the electrons in the plasma and in the beam are zero. Hence, the pressure p_0 of electrons in the plasma and the beam is zero. Applying to the plasma electrons the continuity equation (6.6) and the Euler equation (6.15), we derive equations for the amplitudes of the plasma parameters following from the first two equations of the system (10.8) at $p' = 0$. Elimination of w' from these equations yields

$$N_e' = -ik\frac{eE'N_0}{m\omega^2} \qquad (10.28a)$$

We can obtain the expression for the amplitude of oscillations of the electron density N_b' in the beam in a similar way writing the electron density in the beam as $N_b + N_b' \exp[i(kx - \omega t)]$ and the velocity of the electrons in the beam as $u + w_b \exp[i(kx - \omega t)]$ where the x axis is

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* Langmuir studied the equalizing of the temperature of the beam of electrons ejected from a cathode surface and the temperature of electrons of the gas-discharge plasma which this electron beam penetrates. He found that equalization occurred at much smaller distances from the cathode than the calculations suggested assuming that deceleration of the electrons in the beam is due to their scattering on the charged particles in the plasma. This phenomenon was called the Langmuir paradox.

parallel to the velocity of the beam:

$$N_b' = -ik\frac{eE'N_b}{m\omega^2} \qquad (10.28\text{b})$$

Similar to the last equation of the system (10.8), Poisson's equation (3.2) yields the following equation for the amplitudes of the system's parameters:

$$ikE' = -4\pi e(N_e' + N_b') \qquad (10.28\text{c})$$

Eliminating from the system of equations (10.28) the amplitudes N_e', N_b', and E', we obtain the following dispersion relation:

$$1 = \frac{\omega_0^2}{\omega^2} + \frac{\omega_0^2}{(\omega - ku)^2}\frac{N_b}{N_0} \qquad (10.29)$$

Here $\omega_0 = (4\pi N_0 e^2/m)^{1/2}$ is the frequency of plasma oscillations. When the density of the beam electrons is zero ($N_b = 0$), Eq. (10.29) reduces to Eq. (10.9) where the electron temperature is taken to be zero.

If the phase velocity of the plasma waves ω/k is equal to the velocity u of the electron beam, the interaction between them is the strongest. Let us analyze this case. Since the density of the beam electrons N_b is small compared to the density N_0 of the plasma electrons, the frequency of the plasma oscillations is close to the plasma frequency ω_0 of the plasma. Hence, we shall consider the waves with a wave number $k = \omega/u$, which have the most effective interaction with the electron beam. We write the frequency of these oscillations as $\omega = \omega_0 + \delta$ and insert it into Eq. (10.29). Expanding the result in a series of the parameter δ/ω_0, we obtain

$$\delta = \omega_0 \left(\frac{N_b}{2N_0}\right)^{1/3} \exp\left(\frac{2\pi i n}{3}\right)$$

where n is an integer. We see that $|\delta|/\omega_0 \sim (N_b/N_0)^{1/3} \ll 1$, that is, the above expansion is valid.

If the imaginary component of the frequency, which is equal to the imaginary component of δ, is negative, the wave is attenuated; if it is positive, the wave is amplified. The maximum value of the amplification factor is given by ($n = 1$)

$$-\gamma = \frac{\sqrt{3}}{2}\left(\frac{N_b}{2N_0}\right)^{1/3}\omega_0 = 0.69\left(\frac{N_b}{N_0}\right)^{1/3}\omega_0 \qquad (10.30)$$

The amplitude N_b' varies with time as exp (γt); this result is valid if the plasma oscillations are small and do not affect the properties of the plasma. This type of instability is known as the beam-plasma instability.

The Buneman instability. Let us consider instability of another type which develops if the mean velocity of electrons differs from the mean velocity of ions. Let us formulate the problem. All the plasma ions are at rest and all the electrons travel with a velocity u with respect to the ions. The plasma is quasineutral, that is, the densities of the electrons and ions are equal. We have to determine the maximum amplification factor of plasma oscillations. The electron beam is decelerated owing to the transfer of energy from the beam to the plasma oscillations.

With this formulation the problem is equivalent to the preceding one. In both problems there is an electron beam penetrating the plasma so that the dispersion relation can be derived in a similar way. Denoting the ion mass as M and taking into account that the ion density is equal to the electron density, we obtain the dispersion relation

$$1 = \frac{m}{M}\frac{\omega_0^2}{\omega^2} + \frac{\omega_0^2}{(\omega - ku)^2} \qquad (10.31)$$

instead of (10.29). Tending the ratio m/M to zero, we obtain the following dispersion relation: $\omega = \omega_0 + ku$. Hence, we can write the frequency of the plasma oscillations as

$$\omega = \omega_0 + ku + \delta$$

Substituting this frequency into Eq. (10.31) and expanding the result in a series of the small parameter δ/ω_0, we obtain

$$\frac{2\delta}{\omega_0} = \frac{m}{M}\frac{\omega_0^2}{(\omega_0 + ku + \delta)^2}$$

The electron beam has the strongest interaction with the wave whose wave number $k = -\omega_0/u$. For this wave

$$\delta = \left(\frac{m}{2M}\right)^{1/3}\omega_0 \exp\left(\frac{2\pi i n}{3}\right)$$

where n is an integer. The highest amplification factor corresponds to $n = 1$ and is given by

$$-\gamma = \operatorname{Im} \delta = \frac{\sqrt{3}}{2} \left(\frac{m}{2M}\right)^{1/3} \omega_0 = 0.69 \left(\frac{m}{M}\right)^{1/3} \omega_0 \qquad (10.32)$$

Note that the frequency of oscillation is of the order of the attenuation factor. This type of instability of the electron beam due to the interaction with the plasma electrons is known as the Buneman instability.

Hydrodynamic instabilities. The types of instability discussed above are the so-called kinetic instabilities for which the amplification of oscillations is due to the differences in the character of motion of various groups of particles. The development of the oscillations ultimately results in a change in the velocity distribution of the charged particles. Another type of instabilities is known as the hydrodynamic instabilities. The development of hydrodynamic instabilities involves a displacement of the plasma regions and results, finally, in a variation of the spatial configuration of the plasma. We shall analyze the simplest type of the hydrodynamic instability, namely, the instability of the pinch.

Let us consider the stability of the pinch with respect to the so-called "sausage" instability. This instability changes the radius of the pinch but leaves the axial symmetry of the pinch conserved. We have to find under what conditions an accidental distortion of the pinch will not develop further. Let us assume that the distortion of the pinch results only in a slight curving of the magnetic lines of force, that is, the radius of curvature of the magnetic lines of force is considerably larger than the radius of the pinch. According to Eq. (9.23), in the plasma region the following equation is then satisfied:

$$p + \frac{H^2}{8\pi} = \text{const}$$

Let us analyze the variation of the parameters of the pinch due to the variation of its radius. The total current and magnetic flux through the cross section of the pinch must be conserved in the process. The electric current is $I_z = caH_\varphi/2$ where a is the pinch radius, and H_φ is the axial magnetic field strength. The condition $\delta I_z = 0$ yields $(\delta a/a) + (\delta H_\varphi/H_\varphi) = 0$ where δa is the variation of the pinch radius, and δH_φ is the variation of the axial magnetic field at the pinch surface on the outside of the plasma. The longitudinal magnetic field is frozen into the plasma so that a displacement of the plasma elements does not change the magnetic flux through them. The condition of conservation of the magnetic flux $\Phi_z = \pi a^2 H_z$ yields $(2\delta a/a) + (\delta H_z/H_z) = 0$ (here H_z is the longitudinal magnetic field inside the plasma). Hence, $(\delta H_z/H_z) = 2(\delta H_\varphi/H_\varphi)$. The variation of the magnetic field pressure inside the plasma $\delta(H_z^2/8\pi) = H_z\delta H_z/(4\pi)$, and the variation of the magnetic pressure outside the plasma is $H_\varphi \delta H_\varphi/(4\pi) = H_\varphi^2 \delta H_z/(8\pi H_z)$. It may be seen that if

$$H_z^2 \geqslant H_\varphi^2/2 \qquad (10.33)$$

holds true, the additional internal magnetic field pressure produced by the above distortion of the pinch is larger than the additional outside magnetic field pressure. When the condition (10.33) is satisfied, the pinch is stable with respect to displacements of the sausage type.

11 Radiation in Gas

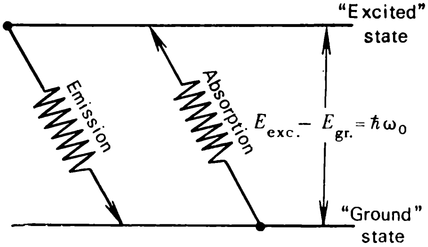
Interaction between radiation and gas. Let us discuss the interaction between an electromagnetic field and an atomic medium, which gives rise to transitions between atomic states. Table 4 presents a summary of one-photon processes due to interaction between the radiation field and atomic systems. This interaction is weak; the small parameter characterizing the weakness of this interaction is the fine structure constant $e^2/\hbar c = 1/137 \ll 1$, which is the ratio between the characteristic velocity of the valence electrons and the velocity of light. Another parameter which proves to be small in reality is the ratio between the electromagnetic radiation field and a certain characteristic quantity. Since these parameters are small, the transitions in the atomic systems involving absorption or emission of photons are slow compared to the characteristic atomic times or to similar atomic processes.

Table 4 Elementary Processes of Interaction Between Radiation and Atomic Systems

Elementary process	Process scheme
Excitation of an atomic particle due to photon absorption	$\hbar\omega + A \rightarrow A^*$
Spontaneous de-excitation of an excited atomic particle	$A^* \rightarrow A + \hbar\omega$
Stimulated emission of a photon	$\hbar\omega + A^* \rightarrow 2\hbar\omega + A$
Photoionization of an atom or molecule	$\hbar\omega + A \rightarrow A^+ + e$
Photodecay of a negative ion	$\hbar\omega + A^- \rightarrow A + e$
Photodissociation of a molecule	$\hbar\omega + AB \rightarrow A + B$
Photorecombination of an ion and electron	$e + A^+ \rightarrow A + \hbar\omega$
Radiative attachment of an electron to an atom	$e + A \rightarrow A^- + \hbar\omega$
Photorecombination of atoms	$A + B \rightarrow AB + \hbar\omega$
Bremsstrahlung of electrons in collisions with an atom or ion	$e + A^+ \rightarrow e + A^+ + \hbar\omega$

For instance, the lifetime of an excited atomic particle, τ, which is determined by its de-excitation in the absence of external fields, is considerably larger than the characteristic atomic times. The reciprocal quantity $1/\tau$ (the frequency of spontaneous de-excitation), apart from the atomic parameters, depends on the factor $(e^2/\hbar c)^3$ and, hence, is lower by at least six orders of magnitude than the frequency of the emitted radiation. Since the interaction between the radiation field and the atomic system is weak, we can simplify the description of the radiative transitions. In particular, we can represent the electromagnetic field as made up of noninteracting photons and neglect all two-photon transitions. The times of these transitions are large compared to the times of one-photon transitions (that is, transitions resulting in emission or absorption of one photon).

FIG. 19. Radiative transition between two levels.



Spontaneous and stimulated emission. Let us analyze the transitions resulting in emission or absorption of a photon between two states of an atom or molecule, the states denoted by the subscripts "gr." (ground) and "exc." (excited), respectively (Fig. 19). (The names given to the states are purely conventional.) Assume that the gas contains n_ω photons in a definite state. The number of these photons can increase due to emission caused by transition of the atom from the "excited" state to the "ground" state or can decrease due to absorption caused by the reverse transition. Assuming the one-photon transition, we write the probability of photon absorption by one atom per unit time:

$$W(\text{gr.}, n_\omega \rightarrow \text{exc.}, n_\omega - 1) = An_\omega \qquad (11.1)$$

Here we made use of the fact that no transitions occur in the absence of photons ($n_\omega = 0$) and only one-photon transitions take place (that is, the higher-power terms in n_ω can be neglected). The quantity A is independent of the electromagnetic field and is determined only by the atomic parameters.

The probability of the atomic transition with emission of a photon is given by

$$W\,(\text{exc.},\ n_\omega \to \text{gr.},\ n_\omega + 1) = 1/\tau + Bn_\omega \qquad (11.2)$$

Here $1/\tau$ is the rate of spontaneous emission of the atom which occurs in the absence of external fields, and the coefficient B refers to the radiation stimulated by the external electromagnetic field. Both parameters depend only on the properties of the atom. The coefficients A and B are known as the Einstein coefficients.

Let us find a relationship for the quantities A, B, and $1/\tau$ in the case of thermodynamic equilibrium between the radiation and the atomic system. The densities of the atoms at the "excited" and "ground" states are related by the Boltzmann distribution (2.9):

$$N_{\text{exc.}} = \frac{g_{\text{exc.}}}{g_{\text{gr.}}} N_{\text{gr.}} \exp\left(-\frac{\hbar\omega}{T}\right) \qquad (11.3)$$

where $\hbar\omega$ is the energy difference between these states, and $g_{\text{exc.}}$ and $g_{\text{gr.}}$ are the statistical weights of the "excited" and "ground" states of the atom. The mean number of photons in a given state can be found from Planck's distribution (2.21):

$$\bar{n}_\omega = [\exp(\hbar\omega/T) - 1]^{-1} \qquad (11.4)$$

Under thermodynamic equilibrium the number of emission transitions per unit time must be equal to the number of absorption transitions per unit time. We write this equality for a unit volume:

$$N_{\text{gr.}} W\,(\text{gr.},\ \bar{n}_\omega \to \text{exc.},\ \bar{n}_\omega - 1)$$

$$= N_{\text{exc.}} W\,(\text{exc.},\ \bar{n}_\omega \to \text{gr.},\ \bar{n}_\omega + 1)$$

Using Eqs. (11.1) and (11.2), we obtain

$$N_{\text{gr.}} A\bar{n}_\omega = N_{\text{exc.}} (1/\tau + B\bar{n}_\omega) \qquad (11.5)$$

Equations (11.5), (11.3), and (11.4) yield $A = g_{\text{exc.}}/(g_{\text{gr.}}\tau)$ and $B = 1/\tau$. Now we can write Eqs. (11.1) and (11.2) in the following form:

$$W\,(\text{gr.},\ n_\omega \rightarrow \text{exc.},\ n_\omega - 1) = \frac{g_{\text{exc.}}}{g_{\text{gr.}}\tau}\, n_\omega, \qquad (11.6)$$

$$W\,(\text{exc.},\ n_\omega \rightarrow \text{gr.},\ n_\omega + 1) = \frac{1}{\tau} + \frac{n_\omega}{\tau} \qquad (11.7)$$

The second term in Eq. (11.7) is the so-called stimulated radiation, which is of fundamental importance.

Broadening of spectral lines. Let us determine the energy distribution of the photons emitted in the transition between the two atomic states. We introduce the frequency distribution function a_ω of photons: $a_\omega d\omega$ is the probability that the frequency of the emitted electromagnetic radiation is in the interval from ω to $\omega + d\omega$. Conservation of energy indicates that the energy of the emitted photon is $\hbar\omega_0 = E_{\text{exc.}} - E_{\text{gr.}}$. Therefore, the width of the photon frequency distribution function is small compared to the frequency of the emitted photon ω_0. We shall determine the form of the frequency distribution function a_ω for photons in various specific cases.

The Doppler broadening. Let us analyze the simplest mechanism of spectral line broadening which is due to the motion of the emitting atoms. The electromagnetic wave emitted by a moving atom with a frequency ω_0 is received by a stationary detector as a wave with a frequency ω; the frequencies ω and ω_0 are related by the Doppler equation

$$\omega = \omega_0(1 + v_x/c) \qquad (11.8)$$

where v_x is the velocity of the atom in the direction of propagation of the emitted wave, and c is the velocity of light.

If we assume the Maxwellian velocity distribution (2.15) for the atoms and the Doppler relationship (11.8) between the atom velocity and the frequency of the emitted photon, we can transform the equation $a_\omega d\omega = \varphi(v_x)\,dv_x$ into the following expression for the profile of the spectral line:

$$a_\omega = \frac{1}{\omega_0}\left(\frac{Mc^2}{\pi T}\right)^{1/2} \exp\left[-\frac{Mc^2}{T}\frac{(\omega-\omega_0)^2}{\omega_0^2}\right] \qquad (11.9)$$

The broadening of the spectral line due to the motion of the emitting particles is known as the Doppler broadening.

Broadening due to finite lifetimes of states. Let us analyze the broadening of spectral lines due to the fact that the states between which the transition occurs have finite lifetimes. The lifetimes of the states can be determined by the transitions owing to collisions as well as by the radiative transition being considered.

The amplitude of the electromagnetic field (that is, the strength of the electric or magnetic field of the electromagnetic wave) is proportional to the product of the wave functions $\psi^*_{\text{exc.}}(t)$ and $\psi_{\text{gr.}}(t)$ of the "excited" and "ground" states the transition between which produces the photon. The steady-state time dependence of the wave function is given by $\exp(-i\mathscr{E}_{\text{gr. exc.}}t/\hbar)$ where $\mathscr{E}_{\text{gr.}}$ and $\mathscr{E}_{\text{exc.}}$ are the energies of the "ground" and "excited" states. Hence, we obtain the following time dependence for the amplitude of the electromagnetic field:

$$f(t) \sim \exp(i\omega_0 t) \qquad (11.10)$$

where $\omega_0 = (\mathscr{E}_{\text{exc.}} - \mathscr{E}_{\text{gr.}})/\hbar$. It may be seen that the frequency distribution of the emitted photons in this case is given by Eq. (11.7) since the photon energy is determined by the conservation of energy.

Now assume that the states of the atomic system between which the transition occurs have finite lifetimes. Let τ_k be the lifetime of the k state. Hence, the probability $|\psi_k|^2$ that the system is in state k satisfies the equation of radioactive decay

$$\frac{d|\psi_k|^2}{dt} = -\frac{1}{\tau_k}|\psi_k|^2$$

and the probability of being in a given state decreases as $|\psi_k|^2 = \exp(-t/\tau_k)$ (at the beginning the system is in the given state). Then the time dependence of the wave function is given by $\psi_k \sim \exp[-i\mathscr{E}_k t/\hbar - t/(2\tau_k)]$, and we obtain the following expression for the amplitude of the electromagnetic field

$$f(t) \sim \exp(i\omega_0 t - \nu t), \quad 2\nu = \tau^{-1}_{\text{gr.}} + \tau^{-1}_{\text{exc.}} \qquad (11.11)$$

instead of Eq. (11.10). Here $\tau_{\text{gr.}}$ and $\tau_{\text{exc.}}$ are the lifetimes of the "ground" and "excited" states. Figure 20*a* gives the field amplitude as a function of time when the field is damped due to the finite lifetimes of the states.

The Fourier transform of the amplitude $f(t)$ yields the frequency distribution for the amplitude of the electromagnetic field:

$$f_\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)\exp(-i\omega t)\,dt \sim [\nu + i(\omega - \omega_0)]^{-1}$$

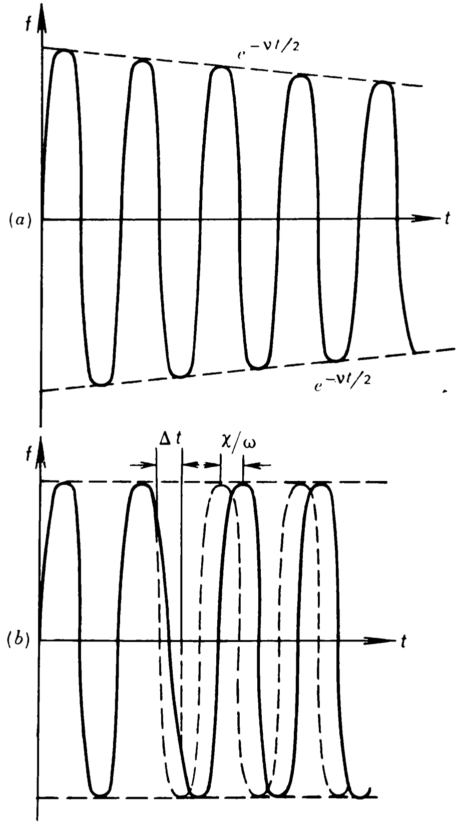


FIG. 20.
The electromagnetic radiation amplitude as a function of time: (a) for finite lifetimes of the states and (b) for a collision between the emitting atom and a gas particle; Δt is the duration of collision, and χ is the phase shift due to collision.

Such parameters as the radiation intensity and the frequency distribution function a_ω of photons are quadratic functions of the field amplitude. We derive the function a_ω by taking into account that $a_\omega \sim |f_\omega|^2$ and using the normalization condition $\int_{-\infty}^{+\infty} a_\omega d\omega = 1$:

$$a_\omega = \frac{\nu}{\pi} \frac{1}{\nu^2 + (\omega - \omega_0)^2} \qquad (11.12)$$

The frequency distribution function (11.12) of the emitted photons is known as the Lorentz profile. If the quantity $\tau = (2\nu)^{-1}$ in Eq. (11.12) is the lifetime of the "excited" state depending on the radiative transition to the "ground" state, then this type of broadening is called the radiative broadening.

Impact broadening of spectral lines. Let us analyze the broadening of the spectral lines due to the interaction between the emitting atom and the surrounding gas particles. First, let us consider the case when the duration of the interaction between the emitting atom and the gas particles is much smaller than the lifetime of the atom states. This means that the emitting atom occasionally collides with a gas particle and the time of collision is much shorter than the time interval between two successive collisions. This type of broadening of spectral lines is known as the impact broadening.

The impact broadening has the following mechanism. The electromagnetic radiation field is produced by the transition of an atom from the "excited" state to the "ground" state and this field is given, up to a phase factor, by Eq. (11.10). When the emitting atom collides with a gas particle, the transition frequency is changed and the general form of the radiation field in this case is shown in Fig. 20*b*.

Assuming that the duration of the collision is small compared to the time interval between two successive collisions, we can write the radiation field as

$$f(t) \sim \exp\left[i\omega_0 t + i \sum_k \chi_k \eta (t - t_k)\right]$$

Here $\eta(x) = 0$ when $x < 0$ and $\eta(x) = 1$ when $x > 0$, and χ_k is the variation of phase following the kth collision. The

Fourier transform of $f(t)$ is

$$f_\omega = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} f(t) \exp(-i\omega t)\, dt$$

$$\sim \sum_k \frac{1-\exp(i\omega_0\tau_k)}{\omega-\omega_0} \exp i \left(\sum_{j<k} \chi_j + \frac{\pi}{2} \right)$$

Here τ_k is the time interval between the kth and the $(k+1)$ st collisions.

To determine the frequency distribution function for photons, let us average it over the phase shifts χ_k occurring due to collisions. Assume that the phase shifts χ_k are random and not small so that we have $\overline{\exp[i(\chi_k - \chi_j)]} = \delta_{jk}$ ($\delta_{jk} = 0$ when $j \neq k$ and $\delta_{jk} = 1$ when $j = k$). Taking into account that $|1 - \exp(ix)|^2 = 2(1-\cos x)$, we find the following expression for the frequency distribution of the emitted photons:

$$a_\omega = \frac{A}{(\omega-\omega_0)^2} \langle 1 - \cos(\omega-\omega_0)\, t \rangle \qquad (11.13)$$

Here A is the normalization factor, t is the time interval between two successive collisions, and the angle brackets denote averaging over time t. Let us express the result of averaging in terms of the mean time τ between two successive collisions.

Denote by $W(t)$ the probability that a collision occurs at the moment t; when $t \to 0$, we have $W(t) = \nu t$. The following equation may be written for the probability W:

$$W(t+t_1) = W(t) + [1 - W(t)]\, W(t_1)$$

Tending t_1 to zero, we obtain the equation $dW/dt = \nu(1-W)$ whose solution is $W(t) = 1 - \exp(-\nu t)$. The probability that a collision occurs in the time interval dt is $(dW/dt)\, dt$; hence, the value of a function $X(t)$ averaged over the time between two successive collisions is given by

$$\langle X \rangle = \int_0^\infty X \frac{dW}{dt} dt$$

Therefore, the mean time between two successive collisions is

$$\tau = \int_0^\infty t \exp(-\nu t)\, \nu\, dt = \frac{1}{\nu}$$

Thus, we have

$$W(t) = 1 - \exp(-t/\tau)$$

and

$$\langle 1 - \cos(\omega - \omega_0) t \rangle = \int_0^\infty e^{-t/\tau} \frac{dt}{\tau} [1 - \cos(\omega - \omega_0) t]$$
$$= \frac{(\omega - \omega_0)^2 \tau^2}{1 + (\omega - \omega_0)^2 \tau^2}$$

Inserting this relation into Eq. (11.13) and taking into account the normalization condition, we find the frequency distribution function for photons

$$a_\omega = \frac{\tau}{\pi} \frac{1}{[(\omega - \omega_0)^2 \tau^2 + 1]} \qquad (11.14)$$

It may be seen that impact broadening also gives rise to the Lorentz profile and Eq. (11.14) is identical to Eq. (11.12) (ignoring the definition of parameter τ). Hence, if the broadening of the spectral line is caused by collisions with gas particles, the shape of the line is given by identical expressions of the type of Eqs. (11.12) or (11.14) irrespective of whether the collisions are elastic or not. The parameter τ in Eqs. (11.12) and (11.14) may be estimated as follows:

$$1/\tau \sim Nv\sigma \qquad (11.15)$$

where N is the density of the gas particles, v is the characteristic velocity of collision between the emitting atom and the gas particle, and σ is the cross section of such a collision.

Assume that the broadening of the spectral lines is due to the interaction between the gas particles and the "excited" state of the emitting atoms and that the contribution of the "ground" state of the emitting atoms into this broadening is small. Let us determine the cross section in Eq. (11.15) for this case assuming the classical motion of the colliding particles. The mechanism of impact broadening indicates that the main contribution to the cross section σ is given by the impact parameter ρ_0 for which the phase shift χ is of the order of unity. Hence, we obtain

$$\sigma \sim \rho_0^2 \qquad (11.16)$$

and for the impact parameter ρ_0 we find that $\int \frac{U(R)\,dt}{\hbar} \sim 1$ where $U(R)$ is the potential of the interaction between the gas particle and the emitting atom in the "excited" state. Since the particles travel in straight lines with this impact parameter, we find that

$$\int \frac{U(R)\,dt}{\hbar} \sim \frac{\rho_0 U(\rho_0)}{\hbar v}$$

Thus, this cross section is of the order of ρ_0^2 and $\rho_0 U(\rho_0)/(\hbar v) \sim \sim 1$. It may be seen that the cross section (11.16) is identical to the total cross section of the collision between the gas particle and the emitting atom in the "excited" state [see Eq. (4.9)].

Let us find the criterion of applicability of the above theory of impact broadening. The time it takes the particle to pass the region of size ρ_0, where scattering occurs, must be much smaller than the time interval between two successive collisions:

$$\frac{\rho_0}{v} \ll (Nv\sigma_t)^{-1}$$

where σ_t is the total cross section of the collisions. Hence, we obtain

$$N\sigma_t^{3/2} \ll 1 \qquad (11.17)$$

Equation (11.17) is the criterion of applicability of the impact broadening theory in the main frequency range of the photon emission.

Statistical broadening theory. Let us analyze another mechanism of spectral line broadening due to the interaction with the gas particles, which is the opposite of the impact broadening. Assume that the gas particles are not able to travel noticeable distances during the characteristic times of broadening development. Hence, the gas particles can be assumed to be stationary and the observed shift of the spectral line is given by

$$\omega-\omega_0=\frac{1}{\hbar}\sum_i V(\mathbf{R}_i) \qquad (11.18)$$

where V is the difference between the potentials of interaction of the emitting atom in the "excited" and "ground" states with the gas particle *, and $\mathbf{R}_i$ is the radius vector of the ith gas particle in a frame of reference with the origin at the nucleus of the emitting atom.

Equation (11.18) expresses the spectral line shift for a given configuration of the interacting gas particles. To find the frequency distribution function for the emitted photons, we must average Eq. (11.18) over all such configurations.

Let $|V(\mathbf{R})|$ monotonically decrease with an increase in the distance between the emitting atom and the interacting gas particles. According to Eq. (11.18), each configuration of the gas particles gives rise to a spectral line shift in the same direction, so that the mean width $\Delta\omega$ of the spectral line is of the same order of magnitude as the mean shift. Since the mean distance between the gas particle is of the order of $N^{-1/3}$ (N is the density of particles), we obtain the following order-of-magnitude estimate according to Eq. (11.18):

$$\Delta\omega\sim\frac{1}{\hbar}V(N^{-1/3}) \qquad (11.19)$$

Let us determine the frequency distribution function of photons at the wing of the line profile. The shift at the wing of the line profile is larger than the mean shift. It is due to interaction with the gas particles which are very close to the emitting atom in the region where they have a low probability of being. The probability for a gas particle to be at a distance from R to $R+dR$ from the emitting atom is $4\pi R^2\,dR\,N$; thus

$$a_\omega d\omega = 4\pi R^2 N\,dR \qquad (11.20a)$$

Here the line shift $\omega-\omega_0$ is

$$\omega-\omega_0=\frac{1}{\hbar}V(R) \qquad (11.20b)$$

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* If the radiative transition removes the atom to the real ground state, the quantity V is practically identical to the interaction potential of the atom in an excited state.

In particular, when $V(R) = CR^{-n}$, Eqs. (11.20) yield the following frequency distribution function of photons at the line wing:

$$a_\omega\, d\omega = \frac{4\pi N C^{3/n}}{n} \frac{d\omega}{(\omega-\omega_0)^{1+3/n}} \qquad (11.21)$$

Let us derive a criterion of applicability for the statistical theory of broadening. The theory holds if the gas particles are not able to change their positions in the time of the order of $(\Delta\omega)^{-1}$ in which the broadening develops. Hence, the criterion is

$$vN^{1/3} \ll \Delta\omega \sim \frac{1}{\hbar} V(N^{-1/3}) \qquad (11.22)$$

where v is the characteristic velocity of collision, and $N^{-1/3}$ is the mean distance between the gas particles or between the emitting atom and the interacting gas particles.

Assume that the broadening is caused only by the interaction with the excited atom state, that is, $V(R)$ in Eqs. (11.16) and (11.22) is identical to the potential $U(R)$ of interaction between the emitting atom in the excited state and the gas particle. Writing the criterion (11.22) as

$$\frac{N^{-1/3} U(N^{-1/3})}{\hbar v} \gg 1$$

and comparing it with Eq. (11.18), we find that for a monotonic potential $\rho_0 \gg N^{-1/3}$. Hence, we can derive the following criterion of applicability for the statistical theory of broadening taking into account that $\sigma_t \sim \rho_0^2$:

$$N\sigma_t^{3/2} \gg 1 \qquad (11.23)$$

Comparing Eqs. (11.23) and (11.17), we see that the impact broadening theory and the statistical broadening theory describe two opposite extreme cases of interaction between the emitting atom and the surrounding gas particles.

The cross sections of emission and absorption of photons. The above mechanisms of spectral line broadening make it possible to estimate the line width due to the motion of the emitting atoms, the finite lifetimes of their excited states, and their interaction with the surrounding gas particles. The frequency distribution function a_ω of the emitted photons, which accounts for the most effective broadening mechanism, can be used for determining the cross sections of absorption and stimulated emission of photons.

Indeed, by definition the cross section of a process is the ratio between the probability of transition per unit time and the incident particle flux. The probability of the stimulated emission per unit time is $Bn_\omega = n_\omega/\tau$, according to Eq. (11.2), where n_ω is the number of the photons of frequency ω in the same state, and τ is the lifetime of the atom state with respect to the radiative transition. Hence, we find that the probability of stimulated emission per unit time due to photons in the frequency range from ω to $\omega + d\omega$ is $(\tau)^{-1} n_\omega a_\omega d\omega$.

The photon flux in this frequency range is $c\, dN_\omega$ where c is the velocity of light, and dN_ω is the density of photons in this frequency range ($dN_\omega = 2n_\omega d\mathbf{k}/(2\pi)^3$ where the factor 2 accounts for two polarizations of photons, and **k** is the wave vector of the photon which is related to the photon frequency as $\omega = kc$). Thus, the photon flux in this frequency range is $n_\omega\, (\omega/\pi c)^2 d\omega$. Hence, the cross section of the stimulated photon emission, which equals the ratio between the frequency of stimulated emission and the incident photon flux, is given by

$$\sigma_{\text{em.}} = \frac{\pi^2 c^2}{\omega^2} \frac{a_\omega}{\tau} \qquad (11.24)$$

The absorption cross section is given by $\sigma_{\text{abs.}} = (\pi c/\omega)^2 A a_\omega$ where A is the Einstein coefficient; according to Eqs. (11.1) and (11.6), $A = g_{\text{exc.}}/(g_{\text{gr.}}\tau)$. Here $g_{\text{exc.}}$ and $g_{\text{gr.}}$ are the statistical weights of the "excited" and "ground" atom states and τ is the time of the respective radiative transition. Then we can easily derive

$$\sigma_{\text{abs.}} = \frac{\pi^2 c^2}{\omega^2} \frac{g_{\text{exc.}}}{g_{\text{gr.}}} \frac{a_\omega}{\tau} \qquad (11.25)$$

Let us find the maximum cross section of absorption. It corresponds to the centre of the minimum-width transition line. According to Eq. (10.12), the distribution function at the centre of the line is $a_\omega = 2\tau/\pi$ and the maximum cross section of absorption is given by

$$\sigma_{\text{abs.}} = 2\pi \frac{g_{\text{exc.}}}{g_{\text{gr.}}} \frac{c^2}{\omega^2} = \frac{g_{\text{exc.}}}{g_{\text{gr.}}} \frac{\lambda^2}{2\pi}$$

where λ is the wavelength of the transition. For instance, when photons are emitted in the visible-light spectrum, this cross section is of the order of 10^{-10}-10^{-9} cm^2, that is, it is larger than the characteristic gas-kinetic cross sections.

The absorption coefficient. Let us define the absorption coefficient for photons, k_ω, by the following equation:

$$\frac{dI_\omega}{dx} = -k_\omega I_\omega \qquad (11.26)$$

Here I_ω is the intensity of radiation of the frequency ω passing through the gas, and x is the distance travelled by the photons in the gas. Assume that absorption or increase of the photon flux is due to transitions between two atomic states. Hence, the absorption coefficient can be written as

$$\begin{aligned} k_\omega &= N_{\text{gr.}}\sigma_{\text{abs.}} - N_{\text{exc.}}\sigma_{\text{em.}} \\ &= N_{\text{gr.}}\sigma_{\text{abs.}}\left(1 - \frac{N_{\text{exc.}} g_{\text{gr.}}}{N_{\text{gr.}} g_{\text{exc.}}}\right) \qquad (11.27) \end{aligned}$$

where $\sigma_{\text{abs.}}$ and $\sigma_{\text{em.}}$ are the cross sections of absorption and emission of photons defined by Eqs. (11.25) and (11.24), and $N_{\text{gr.}}$ and $N_{\text{exc.}}$ are the densities of the atoms in the "ground" and "excited" states.

Equation (11.27) shows that when

$$\frac{N_{\text{exc.}}}{N_{\text{gr.}}} > \frac{g_{\text{exc.}}}{g_{\text{gr.}}} \qquad (11.28)$$

the absorption coefficient k_ω is negative, that is, a photon flux passing through this system is amplified. The situation when the "excited" level has a greater population than the "ground" level is known as the inversion or inverted population of the levels, and the medium where condition (11.28) is satisfied is called the active medium. Active mediums are used for constructing lasers, which are generators of monochromatic radiation.

The conditions of laser operation. A prerequisite of laser operation is the inversion of levels, that is, condition (11.28) must be satisfied. There are various means for producing a greater population at an "excited" level of a certain system than at a "ground" level (for instance, with a gas discharge, optical pumping, or electron beam). This gives rise to an inversion of the levels, but by itself this is not sufficient for laser operation.

The essential condition of laser operation is that the amplification coefficient must not be too small. Let the size of the active medium be L. A mirror is placed at one end of the active medium providing for total reflection of the incident radiation, and means for partial reflection of radiation are provided at the other end of the active medium*. The total probability of the laser radiation escaping or being absorbed at the second end is α. A photon travelling in the active medium is reflected at the ends, and the mean distance it travels until it escapes or is absorbed at the end is $2L/\alpha$. This distance must be larger than the distance $-1/k_\omega$ the photon must travel to give rise to another photon of the same energy. Hence, the condition of the laser operation is given by

$$-k_\omega \geqslant \alpha/2L \tag{11.29}$$

Let us analyze the operation principles of a laser. Assume that the condition (11.29) is satisfied for a certain frequency range at the initial moment when no photons exist in the system. Then the photons of this frequency range accidentally produced in the system and travelling along its axis will stimulate emission of more photons and the photon flux will be amplified. This will give rise to a radiation field in the active medium between the mirrors while the inversion of the levels and the amplification factor will decrease. The condition (11.29) will be satisfied for a narrower frequency range and a further intensification of the radiation field will produce a situation during which only the photons corresponding to the centre of the spectral line with the maximum amplification factor are emitted.

Therefore, the width of the laser line is small; it is determined by the properties of the resonator system, that is, the mirrors. The number n_ω of the photons in the same state

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* This is typically accomplished with a half-silvered mirror, a mirror with a hole at the centre, or a prism.

proves to be large, $n_\omega \gg 1$, and the frequency of the stimulated emission is considerably higher than the frequency of spontaneous emission. Thus, all the emitting (excited) atoms contribute to the useful radiation. Hence, the laser emits a parallel beam of photons in a narrow frequency range.

Propagation of the resonance radiation. Let us discuss the propagation of the resonance radiation, that is, the radiation produced with de-excitation of one of the lowest excited states of the atom. This atom state is known as the resonance state; it is the lowest excited state for which the effective emission transition to the ground state is possible. The photons emitted in this transition are known as the resonance photons.

The free path of the resonance photons in a gas is small since, first, they are absorbed by the ground-state atoms and, secondly, their absorption cross section for the line's centre is greater than the gas-kinetic cross section by a few orders of magnitude. Therefore, propagation of resonance radiation is determined, to a great extent, by re-emission of photons. If the free path of the resonance photon at the line's centre is much smaller than the dimensions of the system, the propagation of the photon cannot be described by the diffusion equation as can be done for the gas particles. A more favourable process than multiple re-emission of the line-centre photon is the radiation of the line-wing photon whose free path is comparable to the dimensions of the system. This possible process is of considerable importance in propagation of the resonance radiation.

Let us discuss the escape of resonance radiation outside the system assuming that the atoms are excited to the resonance state by the collisions with electrons and the escape of radiation does not affect the density $N_{\mathrm{exc.}}$ of the excited atoms. Assume that the free path of the line-centre photons is much smaller than the size of the system L, that is, $k_0 L \gg 1$ where $k_0 = N_{\mathrm{gr.}}\sigma_{\mathrm{abs.}}(\omega_0) - N_{\mathrm{exc.}}\sigma_{\mathrm{em.}}(\omega_0)$ is the absorption coefficient for the line-centre photons.

A statistical equilibrium is established under the above conditions between the atoms and the line-centre photons whose free path is small compared to the size of the system. Let i_ω be the flux of photons of a frequency ω inside the gas. Then the number of photons absorbed in a unit volume per unit time in the frequency range from ω to $\omega + d\omega$ is $i_\omega k_\omega d\omega$ where k_ω is the absorption coefficient (11.27). This number must be equal to the number of photons emitted per unit time in a unit volume in the same frequency range, which is $(N_{\text{exc.}}/\tau) a_\omega d\omega$. Using Eqs. (11.27), (11.24), and (11.25), we obtain

$$i_\omega = \frac{a_\omega N_{\text{exc.}}}{k_\omega \tau} = \frac{\omega^2}{\pi^2 c^2} \left(\frac{N_{\text{gr.}} g_{\text{exc.}}}{N_{\text{exc.}} g_{\text{gr.}}} - 1 \right)^{-1}$$

This photon flux is isotropic and can be detected at any point in the medium which is separated from the boundary by at least the photon free path. The photon flux outside the system is given by

$$I_\omega = \left(\int_0^{\pi/2} i_\omega \cos\theta \, d\cos\theta \right) \left(\int_{-\pi/2}^{+\pi/2} d\cos\theta \right)^{-1} = \frac{1}{4} i_\omega$$

Here θ is the angle between the normal to the gas surface and the direction of photon propagation; in the above equation we have taken into account the fact that the total photon flux outside the system is normal to the system's surface. This leads to the following equation for the photon flux outside the system with a frequency ω:

$$I_\omega = \frac{\omega^2}{4\pi^2 c^2} \left(\frac{N_{\text{gr.}} g_{\text{exc.}}}{N_{\text{exc.}} g_{\text{gr.}}} - 1 \right)^{-1}, \quad k_\omega L \gg 1 \qquad (11.30)$$

When the atoms in the excited and ground states are in thermodynamic equilibrium (2.9), Eq. (11.30) is identical to Eq. (2.26) for black body radiation.

Let us estimate the line width of resonance radiation escaping the system.

For the frequency range in which $k_\omega L \ll 1$, the mean radiation flux escaping outside the gas volume is

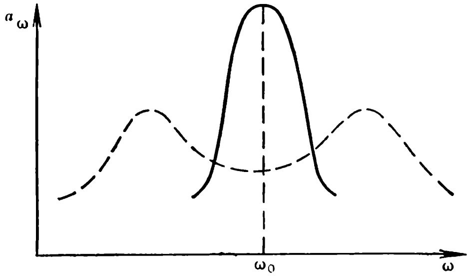
$$I_\omega = \frac{N_{\text{exc.}}}{\tau} a_\omega \frac{V}{S} \qquad (11.31)$$

where V is the gas volume, and S is the surface area of the gas. Equations (11.30) and (11.31) show that the line of the total radiation escaping the system is wider than the line radiated by an individual atom and that its width can be estimated from the following relation:

$$k_\omega L \sim 1 \qquad (11.32)$$

For instance, for the Lorentz profile Eqs. (11.14), (11.25), and (11.27) show that the absorption coefficient is given by $k_\omega = k_0 \Delta\omega_l^2/(\omega - \omega_0)^2$ where k_0 is the absorption coefficient at the line centre, and $\Delta\omega_l$ is the line width. Proceeding from Eq. (11.32), we find that the line width of the radiation escaping outside the system is

FIG. 21. Self-reversal of a spectrum line. The solid line is the line in the absence of re-emission, and the dashed line is the line profile with re-emission.



$$\Delta\omega \approx \Delta\omega_l (k_0 L)^{1/2} \qquad (11.33)$$

if $k_0 L \gg 1$. Similarly, using Eqs. (11.9), (11.25), (11.27), and (11.32), we find that the line width of the radiation escaping outside the system with Doppler broadening is given by

$$\Delta\omega = \Delta\omega_D (\ln k_0 L)^{1/2} \qquad (11.34)$$

where $\Delta\omega_D = \omega_0 [T/(Mc^2)]^{1/2}$ is the line width with Doppler broadening in the absence of re-emission [see Eq. (11.9)].

When $k_\omega L \gg 1$, the radiation of the frequency ω which escapes outside the system is produced by the de-excitation of the excited atoms which are at a distance of the order of $1/k_\omega$ from the system's surface. Hence, into Eq. (11.34) we should insert the temperature of the excited atoms which are in this region of the system. If the electron temperature and, hence, the density of the excited atoms is considerably lower at the boundary than in the bulk of the radiating system, then the emitted line has a profile illustrated in

Fig. 21, that is, a dip appears at the centre of the line. This effect is known as the self-reversal of spectrum lines and is found typically in arc discharges.

12 Plasma of the Upper Atmosphere

The balance equations for the parameters of weakly ionized gas. We have discussed a number of processes of production and decay of charged particles in plasma which are only a fraction of the possible processes involving these particles. What we really have to find is the density of charged particles and the variation of this density in space and time. For instance, the useful parameter in the gas laser is the density of charged particles in the emitting states, in the gas-discharge radiation sources it is the density of the emitting atoms, in the shock tube it is the density of the molecules in the excited vibrational states which determines the vibrational temperature of the molecules, and in the conducting gas it is the density of electrons. To find the parameters needed in a particular problem we have to derive the balance equation (or system of equations) for them and solve it.

The balance equation for the density of particles (or for any other parameter) in plasma is simple in form; it accounts for all the processes producing variations of the density of particles. For instance, Eq. (4.4) is the balance equation for the density of particles in one state when the transitions to other states are due to collisions with particles of another species. Generally, the main problem in deriving the balance equation for a parameter of a plasma is to identify the process which gives rise to a variation of this parameter. If we can correctly identify the principal processes or mechanisms which determine the effect studied or the properties of a given plasma, our problem will be solved. Otherwise, no sophisticated mathematics will help. This is why we have paid attention in this book mostly to those features which make up the physical picture of the process being analyzed rather than to solution of the equations which describe this picture.

Let us now illustrate our analysis by discussing the plasma of the upper atmosphere and showing how the study of elementary processes can yield a qualitative description of the system. We wish to find the distribution of charged and neutral species with height, the composition of the upper atmosphere, and its heat balance. We shall make use of the descriptions of the elementary processes made above and the numerical values of the parameters of this system.

The distribution of particles and temperature in the atmosphere with height. The atmosphere near the earth's surface at sea level consists primarily of molecular nitrogen (78%), molecular oxygen (21%), and the total gas density is $2.7 \times \times 10^{19}$ cm^{-3} (the pressure is 1 atm). Let us find the variation of the gas density with height. We are interested in heights which are small compared to the earth's diameter (12 800 km) so that the earth's surface can be assumed flat. The total flux density of the molecules to the earth's surface is

$$\mathbf{j} = -\mathscr{D}\,\mathrm{grad}\,N + \mathbf{w}N = 0 \qquad (12.1)$$

Here $\mathbf{w} = bM\mathbf{g}$ is the drift velocity of molecules under the action of earth's gravity, N is the density of molecules, $\mathscr{D}$ is the diffusion coefficient of molecules, b is the mobility of molecules, M is the mass of molecules being considered, and $\mathbf{g}$ is the acceleration of gravity. According to the Einstein relation, $\mathscr{D} = bT$ where T is the gas temperature. Hence, we obtain the steady-state ($\mathbf{j} = 0$) distribution of the molecules of a given species:

$$N = N_0 \exp\left(-\int_0^h \frac{Mg\,dz}{T}\right) \qquad (12.2)$$

Here N_0 and N are the densities of molecules at sea level and at height h. If the temperature is independent of height, Eq. (12.2) is identical to the barometric formula (2.14).

Let us make an order-of-magnitude estimate for the drift velocity of heavy species using Eqs. (7.15) and (7.17) and the estimate (7.2) for the diffusion coefficient:

$$w = \frac{Mg}{T}\mathscr{D} \sim \left(\frac{M}{T}\right)^{1/2} \frac{g}{N_m \sigma_g}$$

Here $\sigma_g \sim 10^{-15}$ cm^2 is the gas-kinetic cross section and N_m is the total gas density. Hence, we obtain

$$wN_m \approx 3 \times 10^{13}\ \text{cm}^{-2}\text{s}^{-1} \qquad (12.3)$$

When estimating the temperature gradient, we shall ignore heat transport processes. Hence, the system is adiabatic and $NT^{\gamma-1}$ is constant at all heights so that we have $dN/N = -(\gamma-1)\, dT/T$ where the adiabatic exponent γ for air is 1.4. Equation (12.2) yields $dN/N = -(Mg/T)\, dz$, and we find that $dT/dz = Mg\,(\gamma - 1) \approx 14$ K-km^{-1}. Thus, the atmospheric temperature decreases with height. This is the case up to the atmospheric layers where the energy of solar radiation is absorbed. The value of dT/dz is overestimated since we ignored the heat transport processes, which are determined, primarily, by evaporation and long-wave radiation. It can be readily shown that heat transport due to thermal conduction is inessential. Indeed, the thermal conductivity $\varkappa$ is of the order $v_{\text{thermal}}/\sigma_g$ where $v_{\text{thermal}} \approx 3 \times 10^4$ cm-s^{-1} is the thermal velocity of the gas molecules, and $\sigma_g \approx 10^{-15}$ cm^2 is the gas-kinetic cross section of the collisions between molecules. Hence, the heat flux $q \sim \varkappa\,(dT/dz) \approx 10^{-8}$ W-cm^{-2} due to thermal conduction is considerably smaller than the radiation flux at the earth's surface ($\sim$ 0.1 W-cm^{-2}).

The heat balance of the earth. Let us analyze the transformations of the solar radiation energy. Assume that the sun is an absolutely black body with a surface temperature of 5800 K. Then the radiation flux from the sun's surface is $I_{\text{sun}} = 6.4$ kW-cm^{-2}, the maximum intensity of the photon flux corresponds to the photon energy of the order of an electron volt, and the main part of the flux is in the optical range. The solar radiation flux at the earth's orbit is $I_{\text{earth}} = 0.14$ W-cm^{-2}. The quantity I_{earth} is referred to as the solar constant.

The solar radiation absorbed by the earth must be radiated back to space since during its existence the earth received such an amount of energy that would have been sufficient to evaporate it. Let us consider the mechanisms for returning the solar energy received by the Earth into space. There are two possible mechanisms: the radiation of photons and the escape of high-energy particles. Let us first estimate the energy flux due to the latter mechanism.

It is obvious that only those particles whose velocity component perpendicular to the earth's surface is higher than the escape velocity $v_{\text{esc.}} = (2gR_{\text{earth}})^{1/2} \approx 1.1 \times \times 10^6$ cm-s^{-1} can escape from the earth's gravitational field ($R_{\text{earth}} = 6400$ km is the earth's radius). The atmospheric particles whose density N_m satisfies the condition $N_m \sigma_g L \leqslant \leqslant 1$ can escape without collisions (here $\sigma_g \sim 10^{-15}$ cm^2 is the gas-kinetic collision cross section, and $L = Mg/T \approx 10$ km is the distance at which a noticeable variation of the atmospheric density occurs). Hence, we obtain $N_m \approx 10^9$ cm^{-3} and the energy flux due to the particles escaping from this layer is $I \sim N_m v_{\text{esc.}}^3 \times \exp[-Mv_{\text{esc.}}^2/(2T)]$ where T is the gas temperature. The energy flux due to this mechanism is of the order of 0.01 W-cm^{-2} $\times$ $\exp[-Mv_{\text{esc.}}^2/(2T)]$. Since the gas temperature T is much lower than $Mv_{\text{esc.}}^2/2 = 20\,000$ K, this energy flux is considerably lower than the solar energy flux. This means that the solar energy received by the earth cannot be returned to space by escaping high-energy atoms and molecules.

Hence, the solar energy absorbed by the earth is emitted to space as long-wave radiation. The diagram in Fig. 22 illustrates the mechanisms of transformation of energy received by the earth's surface and the atmosphere. In particular, the thermal radiation by the earth's surface amounts to 2.06×10^{14} kW. Assuming that the earth is radiating like a black body, we can estimate from the Stefan-Boltzmann law (2.26) that the mean temperature of the earth's surface is 291 K or 18 °C.

The elemental oxygen in the atmosphere. At high altitudes there occurs effective photodissociation of the atmospheric molecular oxygen:

$$O_2 + \hbar\omega \rightarrow O(^3P) + O(^1D) \qquad (12.4)$$

This process gives rise to radiation absorption known as the Schumann-Runge continuum, in the wavelength range from 1325 to 1759 Å (the photon energy from 6 to 10.3 eV) with the cross sections of the order of 10^{-19}-10^{-17} cm^2. Photodissociation of molecular oxygen changes the composi-

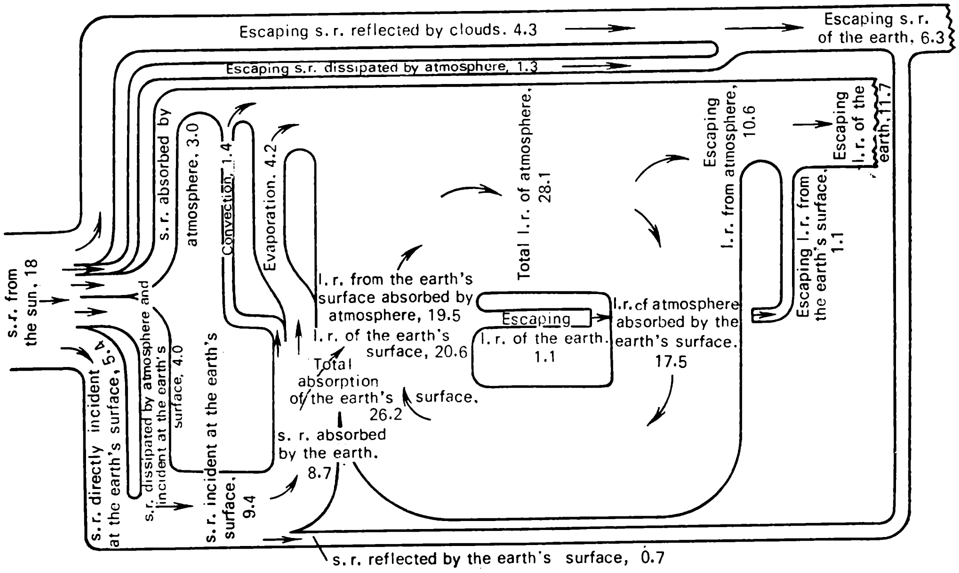


FIG. 22.
The heat balance of the earth's surface and its atmosphere.
The values given in the figure are the mean converted powers in units of 10^{13} kW;
s.r. is the short-wave radiation (the optical and neighbouring ranges);
l.r. is the long-wave radiation (the infrared range).

tion of the atmosphere at high altitudes. The density of elemental oxygen becomes equal to the density of molecular oxygen at the altitude of 100-120 km and to the density of molecular nitrogen at the altitude of 150-200 km. Thus, elemental oxygen is a major component of the upper atmosphere.

First, let us analyze absorption of the short-wave radiation coming to the earth, which produces photodissociation of an oxygen molecule. The balance equation for the intensity I_ω of this radiation can be written as follows:

$$-\frac{dI_\omega}{dz} = -I_\omega \sigma_\omega [O_2]$$

Here z is the altitude, $[O_2]$ is the density of molecular oxygen, and σ_ω is the cross section of photodissociation of an oxygen molecule. Assume that radiation flux is normal to the earth's surface and the density of molecular oxygen varies according to the barometric formula

$$[O_2] = N_0 \exp(-z/L)$$

Here $L = Mg/T \approx 10$ km. The above equation yields

$$I_\omega(z) = I_\omega(\infty) \exp\{-\exp[-(z-z_0)/L]\} \qquad (12.5)$$

where $I_\omega(\infty)$ is the photon flux at frequency ω outside the atmosphere, and the point z_0 is found from the equation $[O_2]_{z=z_0} = (\sigma_\omega L)^{-1}$. Equation (12.5) indicates that the greater part of the photons of this frequency are absorbed at the altitude of about z_0 in the atmospheric layer with a thickness of about L.

The cross secion of photon absorption by an oxygen molecule varies from 10^{-19} to 10^{-17} cm^2 for these photon frequencies so that these photons are absorbed mainly in those atmospheric layers where the molecular oxygen density $[O_2]$ is of the order of 10^{11}-10^{13} cm^{-3}.

Let us verify the validity of the barometric formula for the molecular oxygen density at these altitudes. The characteristic time of photodissociation of the oxygen molecule at the altitudes where $[O_2] \approx 10^{11}$ cm^{-3} and where the 1300-1600 Å photons are mostly absorbed is given by $\tau_{\mathrm{dis.}} \sim \sim \left(\frac{1}{4}\int \sigma_\omega dI_\omega\right)^{-1} \approx 2 \times 10^6$ s (the factor 1/4 is due to the averaging of the solar radiation flux over the earth's surface). The characteristic time in which a molecule passes the above layer (drifts from it) is

$$\tau_{\text{drift}} \sim L/w \sim 3 \times 10^{-8} N_m \text{ cm}^3\text{s} \qquad (12.6)$$

where N_m is the density of particles in the atmosphere. Assuming $[O_2] \approx N_m/4$, we find $\tau_{\text{drift}} \approx 10^4$ s, that is, $\tau_{\text{drift}} \ll \tau_{\text{dis.}}$. Hence, photodissociation cannot violate the barometric formula for the distribution of molecular oxygen.

Let us estimate the density of elemental oxygen in the atmospheric layers where the photodissociation of molecular oxygen occurs. The flow $w\,[O]$ of oxygen atoms drifting from this layer under the action of the earth's gravity is equal to the flow of atoms produced by photodissociation, which equals the flow of photons of the appropriate frequency at this altitude. Using Eq. (12.3), we find the following value of the elemental oxygen density [O] at this altitude (the solar radiation flux giving rise to the photodissociation of molecular oxygen is $I \sim 10^{13}$ cm^{-2}s^{-1}):

$$[O] \sim I/w \sim 0.1\, N_m \qquad (12.7)$$

Hence, [O] is of the order of 10^{12} cm^{-3} at this altitude. The barometric formula may be used for finding the elemental oxygen density at higher altitudes.

The barometric formula can be applied to elemental oxygen at lower altitudes so that its density must increase with decreasing altitude. With increasing elemental oxygen density, there increases the probability of the three-particle process of recombination of oxygen atoms:

$$2O + M \rightarrow O_2 + M, \quad O + O_2 + M \rightarrow O_3 + M \qquad (12.8)$$

Here M is the nitrogen or oxygen molecule. The elemental oxygen density is the highest at the altitude where the rate of recombination of oxygen atoms equals the rate of production of oxygen atoms. Let us estimate the highest elemental oxygen density from this condition:

$$w\,[O]_{\max}/L \sim \mathscr{K}\,[O]^2_{\max} N_m \qquad (12.9)$$

Here $[O]_{\max}$ is the highest density of elemental oxygen, $\mathscr{K} \sim 10^{-33}$ cm^{6}s^{-1} is the rate constant of the three-particle recombination of oxygen atoms, and w is the drift velocity of the oxygen atom towards the earth's surface due to gravity, which is given by Eq. (12.3). Using Eqs. (12.7) and (12.9) and the numerical values of the parameters, we find that $[O]_{\max} \sim 10^{13}$ cm^{-3}.

Charged particles in the upper atmosphere. Photoionization of atoms and molecules in the upper atmosphere produces ions, for instance, N_2^+, O_2^+, N^+, and O^+. The photon flux I_{ion}, with photon energies exceeding the ionization potentials, for these molecules is of the order of 3×10^{10} cm^{-2}s^{-1} in the upper atmosphere, and the respective photoionization cross section is $\sigma_{\text{ion}} \sim 10^{-18}$-$10^{-17}$ cm^2. Equation (12.5) indicates that this photon flux is absorbed mainly in the atmospheric layers where the particle density N_m is of the order of $(\sigma_{\text{ion}}L)^{-1} \sim 10^{11}$-$10^{12}$ cm^{-3}. The primary production of charged particles occurs just in these layers of the upper atmosphere.

Molecular ions decay due to dissociative recombination, and their density can be estimated from the relation $\alpha N_e N_i \sim \sim I_{\text{ion}}/L$ where $\alpha \sim 10^{-7}$ cm^{3}s^{-1} is the rate constant of dissociative recombination, and the electron density N_e is assumed to be of the order of the ion density N_i. Substituting the numerical values, we obtain ($I_{\text{ion}} \sim 3 \times 10^{10}$ cm^{-2}s^{-1}) the following expression:

$$N_i \sim \left(\frac{I_{\text{ion}}}{\alpha L}\right)^{1/2} \sim 5 \times 10^5 \text{ cm}^{-3} \qquad (12.10)$$

Here the characteristic time $\tau_{\text{rec.}}$ of recombination of the molecular ion is of the order of $(\alpha N_e)^{-1} \sim 20$ s while the characteristic time τ_{drift} of the drift of the molecular ion from this layer is of the order of 10^4 s. Therefore, we could ignore the escape of ions to other layers when writing the balance equation for the density of the molecular ions.

Since the upper atmosphere contains electrons, it does not transmit long-wave electromagnetic waves whose frequency is higher than the plasma frequency $\omega_0 = (4\pi N_e e^2/m)^{1/2} = = 3 \times 10^7$ s^{-1}, [see Eqs. (3.9) and (10.9)]. This frequency corresponds to the wavelength $\lambda \approx 60$ m and the electromagnetic waves of lower frequencies will be either damped or reflected from this atmospheric layer known as the ionosphere.

The atomic ions produced by photoionization in the upper atmosphere react with molecules:

$$O^+ + N_2 \rightarrow NO^+ + N + 1.05 \text{ eV}, \qquad (12.11a)$$

$$O^+ + O_2 \rightarrow O_2^+ + O + 1.3 \text{ eV}, \qquad (12.11b)$$

$$N^+ + O_2 \rightarrow NO^+ + N + 5.7 \text{ eV} \qquad (12.11c)$$

The rate constant k of the first reaction is of the order of 10^{-11} cm^{3}s^{-1}, that of the second reaction is of the order of 10^{-12} cm^{3}s^{-1}, and that of the third reaction is of the order of 10^{-10} cm^{3}s^{-1}. Hence, the characteristic times of transformation of atomic ions into molecular ions in this atmospheric layer are of the order of $(kN_m)^{-1} \sim 0.01$-10 s. It may be seen that the transformation of atomic ions into molecular ions occurs faster than the recombination of molecular ions. Therefore, the density of atomic ions at these altitudes is lower than the density of molecular ions. In the atmospheric layer where the density of molecular ions is the highest (the altitude about 100 km), the principal ion species are O_2^+, N_2^+, and NO^+.

Let us analyze the distribution of the atomic ions of oxygen, which constitute the main ion component of the atmosphere at large altitudes, where it consists basically of elemental oxygen and molecular nitrogen. Oxygen ions are produced by photoionization of oxygen atoms, and they disappear in the ion-molecular reaction with nitrogen molecules or drift away due to diffusion. The highest density of oxygen ions corresponds to the equilibrium between these processes:

$$w/L \sim k\,[N_2]$$

Here w is the drift ion velocity (12.3), k is the rate constant of the ion-molecule reaction (12.11a), and $[N_2]$ is the density of nitrogen molecules. Assuming that this atmospheric layer consists of elemental oxygen, we obtain

$$[N_2]\,[O] \sim w\,[O]/(Lk)$$

Assuming, in accordance with Eq. (12.3), $w\,[O] \sim 3 \times 10^{13}$ cm^{-2}s^{-1}, and $L \approx 10$ km and $k \sim 10^{-12}$ cm^{3}s^{-1}, we obtain $[N_2]\,[O] \approx 3 \times 10^{19}$ cm^{-6} for the altitude which corresponds to the highest density of atomic ions. This altitude is about 200 km.

Let us estimate the highest density $(N_i)_{\max}$ of the atomic ions of oxygen from the balance equation for the density of atomic ions $\int \sigma_{\text{ion}}\, dI_{\text{ion}}\ [\text{O}] = k\,[\text{N}_2]N_i$. In the case of photoionization of atomic oxygen, we have $\int \sigma_{\text{ion}}\, dI_{\text{ion}} \sim$ $\sim 2\times 10^{-7}\ \text{s}^{-1}$ and the highest density of the atomic ions of oxygen is of the order of $2\times 10^{5}\ \text{cm}^{-3}\times [\text{O}]/[\text{N}_2] \leqslant$ $\leqslant 10^{6}\ \text{cm}^{-3}$.

At greater altitudes, the density of atomic ions is given by the barometric formula, and it slowly decreases with increasing altitude since the temperature of ions is considerably higher than the gas temperature.

Note that the characteristic times of decay of ions do not exceed 100 seconds. Therefore, the ion density at night is considerably lower than the ion density at daylight. We made all our estimates for daylight conditions. The density of ions at night can be estimated from the relation $\alpha N_i\tau \sim 1$ where τ is the duration of night. Hence, we find that at night $N_i \sim 10^{2}$-$10^{3}\ \text{cm}^{-3}$ in the atmosphere layer where the negative ions are not produced ($\tau \approx 3\times 10^{4}$ s, $\alpha \sim 10^{-7}\ \text{cm}^3\text{s}^{-1}$).

As discussed above, molecular ions are produced mostly in the atmospheric regions where the gas density N_m is of the order of 10^{11}-$10^{12}\ \text{cm}^{-3}$. The molecular ions drift to the regions of higher density under the action of gravity. The density of molecular ions in these regions at daylight can be estimated from the balance equation

$$wN_i/L \sim \alpha N_i^2$$

Since the drift velocity w of ions under the action of gravity is close to the drift velocity (12.3) of atoms and molecules, we find the following estimate:

$$N_iN_m \sim 10^{15}\ \text{cm}^{-6} \qquad (12.12)$$

At smaller altitudes the molecules capture electrons producing negative ions. For instance,

$$e + 2\text{O}_2 \rightarrow \text{O}_2^- + \text{O}_2 \qquad (12.13)$$

The rate constant $\mathcal{K}$ of the process (12.13) is of the order of $10^{-31}\ \text{cm}^6\text{s}^{-1}$, and this process is of significance at altitudes where $wN_e/L \sim \mathcal{K}N_e\,[\text{O}_2]^2$, that is, where the characteristic time of drift of charged particles is equal to the time of electron capture by molecule. Using the formula for ambipolar diffusion and Eq. (12.3) for the drift velocity of electrons, we find that negative ions are predominant among the negative species in the regions where the density of oxygen molecules is

$$[O_2] \gtrsim \left(\frac{w\,[O_2]}{\mathcal{K}L}\right)^{1/3} \approx 10^{13}\ \text{cm}^{-3} \qquad (12.14)$$

In these regions recombination occurs according to the reaction $A^- + B^+ \rightarrow A + B$, and the recombination coefficient α is of the order of 10^{-9} cm^3s^{-1} so that we obtain the following estimate for the ion density instead of Eq. (12.12):

$$N_i N_m \sim 10^{17}\ \text{cm}^{-6} \qquad (12.15)$$

We have considered the plasma of the upper atmosphere to illustrate the techniques used in describing a specific system. The parameters of the plasma have been estimated using the relevant numerical values (the parameters of the solar radiation and the rate constant and the cross sections of elementary processes). Our results do not provide an accurate and detailed description of the system, but they do provide a correct qualitative picture. An incomparably greater amount of work is required to increase the accuracy of description and to account for the details of the process, but the results will be fundamentally the same.

Appendices

Appendix 1 Gas-kinetic Cross Sections and Diffusion Coefficients

The gas-kinetic σ_g cross section for collisions between atoms and molecules are given in units of 10^{-15} cm^2; they are derived from the diffusion coefficients $\mathscr{D}$ using the relation $\sigma_g = T/(\mu \bar{v} N \mathscr{D})$ where μ is the reduced mass of the colliding particles, $\bar{v}$ is the mean relative collision velocity, and N is the density of particles. The diffusion coefficients are given in brackets (in cm^2/s) for a density of 2.689×10^{19} cm^{-3} corresponding to standard conditions (pressure 1 atm and temperature 0°C).

Pair	He	Ne	A	Kr	Xe	H_2	D_2	N_2	O_2	CO	CO_2
He	1.6 (1.5)	2.0 (0.95)	2.9 (0.64)	3.3 (0.55)	3.7 (0.48)	2.2 (1.4)	2.2 (1.1)	3.1 (0.6)	2.9 (0.64)	3.0 (0.62)	3.6 (0.51)
Ne		2.4 (0.45)	3.4 (0.28)	4.0 (0.22)	4.4 (0.19)	2.7 (0.97)	2.4 (0.8)	3.2 (0.32)			4.1 (0.23)
A			5.0 (0.16)	5.6 (0.12)	6.7 (0.095)	3.7 (0.69)	3.2 (0.66)	5.4 (0.16)	5.2 (0.16)	5.3 (0.16)	5.7 (0.14)
Kr				6.7 (0.081)	7.7 (0.064)	4.2 (0.6)	4.3 (0.42)	4.8 (0.16)	5.0 (0.15)	5.9 (0.13)	5.4 (0.12)
Xe					9.0 (0.048)	5.0 (0.5)		7.3 (0.1)	6.9 (0.1)		
H_2						2.7 (1.3)	3.6 (1.2)	3.8 (0.68)	3.7 (0.69)	3.9 (0.65)	4.5 (0.56)
D_2								3.4 (0.54)		3.9 (0.48)	4.5 (0.41)
N_2								5.2 (0.18)	4.1 (0.22)	5.1 (0.18)	6.8 (0.12)
O_2									4.9 (0.18)	4.8 (0.19)	5.9 (0.14)
CO										5.0 (0.19)	6.3 (0.13)
CO_2											7.8 (0.096)

Appendix 2 The Cross Section of Resonance Charge Exchange Between the Positive Ion A^+ and the Respective Atom A

The cross section is given in units of 10^{-15} cm^2, and E is the ion energy for the resting atom; the ion and the atom are in their ground states

	H	He	Li	Be	B	C	N	O
$E=0.1$ eV	6.2	3.5	26	13	9.6	5.3	5	4.8
1 eV	5.0	2.8	22	11	7.4	4.3	3.8	3.5
10 eV	3.8	2.1	18	19	5.8	3.2	3	2.8

	F	Ne	Na	Mg	Al	Si	P	S
$E=0.1$ eV	3.1	3.2	31	19	16.1	8.7	8.1	8.5
1 eV	2.5	2.5	26	16	12.9	6.5	6.5	6.8
10 eV	1.9	1.9	22	13	10.0	4.9	5.0	5.3

	Cl	A	K	Ca	Ti	V	Cr	Mn	Fe
$E=0.1$ eV	4.9	5.5	41	26	22	23	21	20	21
1 eV	3.9	4.5	35	21	19	19	18	16	18
10 eV	3.0	3.6	29	18	15	16	14	13	15

	Co	Ni	Cu	Zn	Ga	Ge	As	Se
$E=0.1$ eV	22	19	19	16	17	9.4	9.8	9.3
1 eV	18	16	16	13	14	7.5	8.0	7.3
10 eV	15	13	13	11	11	6.1	6.3	5.7

	Br	Kr	Rb	Sr	Zr	Nb	Mo	Ag	Cd
$E=0.1$ eV	5.9	7.3	45	30	24	22	21	20	17
1 eV	4.6	5.9	39	25	20	19	17	17	14
10 eV	3.7	4.6	32	21	16	15	14	14	12

	In	Sn	Sb	Te	I	Xe	Cs	Ba
$E=0.1$ eV	19.5	10.7	11.4	10.6	7.0	9.1	5.3	35
1 eV	16	8.7	9.1	8.6	5.6	7.5	45	30
10 eV	13	6.9	7.2	6.8	4.4	6.0	38	25

	Ta	W	Re	Pt	Au	Hg	Tl	Pb	Bi
$E=0.1$ eV	19	18	21	17	15	15	18.6	11	15.4
1 eV	16	15	17	16	14	12	15.1	9.2	12.7
10 eV	13	13	14	13	11	10	12.1	7.3	10.3

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